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ON AFFINE COLLINEATIONS IN A SPACE OF HYPERPLANES

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The geometry of paths initiated by L. P. Eisenhart and O. Veblen has been generalized by J. Douglas to the general geometry of paths and further to the geometry of K-spreads.

In a space of (n—1)-spreads, that is to say, in a space of hyperplanes, the elements of the space are points (x^i) and hyperplane elements (u_i) attached with each point. Thus, the geometry of hyperplanes can be studied in the form dual to the general geometry of paths. This was done recently by K. Yano and the present author [8]*.

The purpose of the present paper is to study affine collineations in such a space and to generalize some of theorems by K. Yano [7]. Corresponding problems in a space of K-spreads have been studied by R. S. Clark [1], E. T. Davies [2] and Buchin Su [3, 4, 5, 6].

We shall use, throughout the paper, the same notations as those in our paper cited above.

1. In an N-dimensional space of system of hyperplanes, referred to a coordinate system (x^i), we consider an infinitesimal point transformation:

$$\bar{x}^i = x^i + \xi^i(x)dt \quad (a,b,c,\cdots\cdots,i,j,k,\cdots\cdots=1,2,\cdots\cdots,N), \tag{1.1}$$

where $\xi^i(x)$ is a contravariant vector field and dt an infinitesimal constant. When there is given a geometric object $\Omega(x,u)$, if we regard (1.1) as a transformation of coordinates of the space, the components $\bar{\Omega}(\bar{x},\bar{u})$ of the object in the new coordinate system will be calculated from the old components $\Omega(x,u)$ by (1.1). Then, the Lie-derivative $X\Omega$ of $\Omega(x,u)$ with respect to $\xi^i(x)$ is defined by the equation

$$D\Omega = \Omega(\bar{x},\bar{u}) - \bar{\Omega}(\bar{x},\bar{u}) = X\Omega dt, \tag{1.2}$$

neglecting the terms in dt higher than of the first order.

From this definition, we can find the Lie-derivatives of various quantities of the space. To express these Lie-derivatives in tensor forms, we define the covariant derivatives of a scalar f and that of a tensor field $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ by

$$f;j = f,j + f/^a\Gamma_{aj}, \tag{1.3}$$

$$T^{i_1\cdots i_r}_{j_1\cdots j_s};k = T^{i_1\cdots i_r}_{j_1\cdots j_s},k + T^{i_1\cdots i_r}_{j_1\cdots j_s}/^a\Gamma_{ak} + \sum_{p=1}^{r} T^{i_1\cdots i_{p-1}a\,j_{p+1}\cdots\cdots i_r}_{j_1\cdots\cdots\cdots\cdots j_s}\Gamma^{i_p}_{ak} - \sum_{p=1}^{s} T^{i_1\cdots\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}a\,j_{p+1}\cdots j_s}\Gamma^{a}_{j_pk} \tag{1.4}$$

respectively, where a comma followed by an index denotes the partial differentiation with respect to x^i and vertical stroke that with respect to u_i.

Thus, the Lie-derivatives of a scalar f, of a tensor field $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ and of the compo-

* See the Bibliography at the end of the paper.

nents of affine connection Γ^{i}_{jk} are given by

$$Xf=f_{;a}\xi^{a}-f/^{a}\xi^{b}_{;a}u_{b}, \tag{1.5}$$

$$XT^{i_1\cdots i_r}_{j_1\cdots j_s}=T^{i_1\cdots i_r}_{j_1\cdots j_s;a}\xi^{a}-\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots j_s}\xi^{i_p}_{;a}$$
$$+\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}\xi^{a}_{;j_p}-T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}\xi^{b}_{;a}u_{b} \tag{1.6}$$

and

$$X\Gamma^{i}_{jk}=\xi^{i}_{;j;k}+R^{i}_{jkl}\xi^{l}-\Gamma^{i}_{jk}/^{a}\xi^{b}_{;a}u_{b} \tag{1.7}$$

respectively. We shall remark here that the Lie-derivatives of these quantities are all components of tensors and that $Xu_{j}=0$.

We can see that the Ricci formulae for an arbitrary tensor field $T^{i_1\cdots i_r}_{j_1\cdots j_s}$:

$$T^{i_1\cdots i_r}_{j_1\cdots j_s;k;l}-T^{i_1\cdots i_r}_{j_1\cdots j_s;l;k}=\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots j_s}R^{i_p}_{akl}$$
$$-\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}R^{a}_{j_pkl}+T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}R_{akl}, \tag{1.8}$$

$$T^{i_1\cdots i_r}_{j_1\cdots j_s;k}/^{l}-T^{i_1\cdots i_r}_{j_1\cdots i_s}/^{l}{}_{;k}$$
$$=\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots j_s}\Gamma^{i_p}_{ak}/^{l}-\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}\Gamma^{a}_{j_pk}/^{l} \tag{1.9}$$

and the generalized Bianchi identities:

$$(R^{i}_{jkl;m}-\Gamma^{i}_{jk}/^{a}R_{alm})+cyc\ (k,l,m)=0 \tag{1.10}$$

hold good, where $cyc(k,l,m)$ denotes two terms obtained from the first by cyclic permutation of indices k,l and m.

By suitable use of the above formulae, we get the following formulae for an arbitrary tensor giving the relations between the Lie-operation and the covariant derivation or partial differentiation with respect to u_{i}:

$$(XT^{i_1\cdots i_r}_{j_1\cdots j_s})_{;k}-XT^{i_1\cdots i_r}_{j_1\cdots j_s;k}=-\sum_{p=1}^{r}T^{i_1\cdots i_{p-1}ai_{p+1}\cdots i_r}_{j_1\cdots\cdots\cdots j_s}X\Gamma^{i_p}_{ak}$$
$$+\sum_{p=1}^{s}T^{i_1\cdots\cdots\cdots i_r}_{j_1\cdots j_{p-1}aj_{p+1}\cdots j_s}X\Gamma^{a}_{j_pk}-T^{i_1\cdots i_r}_{j_1\cdots j_s}/^{a}X\Gamma^{b}_{ak}u_{b}, \tag{1.11}$$

$$(XT^{i_1\cdots i_r}_{j_1\cdots j_s})/^{l}-XT^{i_1\cdots i_r}_{j_1\cdots j_s}/^{l}=0 \tag{1.12}$$

and the similar formulae for the components of affine connection:

$$(X\Gamma^{i}_{jk})_{;l}-(X\Gamma^{i}_{jl})_{;k}=XR^{i}_{jkl}-\Gamma^{i}_{jk}/^{a}X\Gamma^{b}_{al}u_{b}+\Gamma^{i}_{jl}/^{a}X\Gamma^{b}_{ak}u_{b}, \tag{1.13}$$

$$(X\Gamma^{i}_{jk})/^{l}-X\Gamma^{i}_{jk}/^{l}=0. \tag{1.14}$$

The above formulae are frequently used in the latter discussion.

2. If we consider, in the space, an r-parameter continuous group G_{r} whose symbols

are given by $X_\mu f = f_{,a}\xi^a_\mu(x)$ $(\lambda, \mu, \nu, \cdots\cdots = 1, 2, \cdots\cdots, r)$, then there exist the relations

$$(X_\mu X_\nu) f = C_{\mu\nu}{}^\lambda X_\lambda f, \tag{2.1}$$

where $C_{\mu\nu}{}^\lambda$ are constants of structure of the group. From (2.1), we have

$$\xi^a_\mu \xi^i_{\nu,a} - \xi^a_\nu \xi^i_{\mu,a} = \xi^i_{\nu;a}\xi^a_\mu - \xi^i_{\mu;a}\xi^a_\nu = C_{\mu\nu}{}^\lambda \xi^i_\lambda,$$

from which, by the definition of the Lie-derivatives, we get

$$X_\mu \xi^i_\nu = C_{\mu\nu}{}^\lambda \xi^i_\lambda. \tag{2.2}$$

Now, consider r infinitesimal transformations of the space

$$\bar{x}^i = x^i + \xi^i_\mu(x)dt, \tag{2.3}$$

then we have, by straightforward calculations:

$$(X_\mu X_\nu) f = f_{;a} X_\mu \xi^a_\nu - f/^a \ (X_\mu \xi^b_\nu)_{;a} u_b, \tag{2.4}$$

$$(X_\mu X_\nu) T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = T^{i_1 \cdots i_r}_{j_1 \cdots j_s ;a} X_\mu \xi^a_\nu - \sum_{p=1}^{r} T^{i_1 \cdots i_{p-1} a i_{p+1} \cdots i_r}_{j_1 \cdots \cdots \cdots j_s} (X_\mu \xi^{i_p}_\nu)_{;a}$$
$$+ \sum_{p=1}^{s} T^{j_1 \cdots \cdots \cdots i_r}_{i_1 \cdots j_{p-1} a j_{p+1} \cdots j_s} (X_\mu \xi^a_\nu)_{;j_p} - T^{i_1 \cdots i_r}_{j_1 \cdots j_s}/^a (X_\mu \xi^b_\nu)_{;a} u_b, \tag{2.5}$$

$$(X_\mu X_\nu) \Gamma^i_{jk} = (X_\mu \xi^i_\nu)_{;j;k} + R^i_{jkl}(X_\mu \xi^l_\nu) - \Gamma^i_{jk}/^a (X_\mu \xi^b_\nu)_{;a} u_b. \tag{2.6}$$

Thus, we have

Theorem 2.1. If we apply the operators $(X_\mu X_\nu)$ *to an arbitrary scalar, tensor and the components of affine connection, then we get their Lie-derivatives with respect to the vectors* $X_\mu \xi^i_\nu$.

If the $\xi^i_\mu(x)$ are r vectors defining an r-parameter group G_r then (2.4), (2.5) and (2.6) are reduced to

$$(X_\mu X_\nu) f = C_{\mu\nu}{}^\lambda X_\lambda f, \tag{2.7}$$

$$(X_\mu X_\nu) T^{i_1 \cdots i_r}_{j_1 \cdots j_s} = C_{\mu\nu}{}^\lambda X_\lambda T^{i_1 \cdots i_r}_{j_1 \cdots j_s} \tag{2.8}$$

and

$$(X_\mu X_\nu) \Gamma^i_{jk} = C_{\mu\nu}{}^\lambda X_\lambda \Gamma^i_{jk}, \tag{2.9}$$

respectively, and we have

Theorem 2.2. If r *vectors* $\xi^i_\mu(x)$ *define an r-parameter group, then the expressions similar to* (2.1) *for an arbitrary scalar, tensor and the components of affine connection hold.*

3. In a space of system of hyperplanes, a point transformation in the space with the finite equations

$$\bar{x}^i = \bar{x}^i(x^1, x^2 \cdots\cdots, x^N) \qquad \left|\frac{\partial \bar{x}}{\partial x}\right| \neq 0 \tag{3.1}$$

is said to be a collineation, if it transforms each hyperplane into a hyperplane. Under this point transformation, the equations of hyperplanes

$$u_i\,dx^i = 0 \qquad du_j - \Gamma_{jk}(x,u)dx^k = 0 \qquad (u_i = \partial f/\partial x^i)$$

are transformed into

$$\bar{u}_i\,d\bar{x}^i = 0 \qquad d\bar{u}_j - \Gamma_{jk}(\bar{x},\bar{u})d\bar{x}^k = 0 \qquad (\bar{u}_i = \partial f/\partial \bar{x}^i). \tag{3.2}$$

On the other hand, from (3.1) regarded as a transformation of coordinates, the equations of hyperplanes are reducible to

$$\bar{u}_i\,d\bar{x}^i = 0 \qquad d\bar{u}_j - \bar{\Gamma}_{jk}(\bar{x},\bar{u})d\bar{x}^k = 0 \qquad (\bar{u}_i = \partial f/\partial \bar{x}^i). \tag{3.3}$$

We shall call the transformation (3.1) an affine collineation, if the equations (3.2) and (3.3) are equivalent under a suitable change of constant factor of f defining the hyperplanes. Thus, for affine collineation, we must have $\Gamma_{jk}(\bar{x},\bar{u}) - \bar{\Gamma}_{jk}(\bar{x},\bar{u}) = 0$, from which, differentiating these equations with respect to $\bar{u}_i$, we have

$$\Gamma^{\,i}_{jk}(\bar{x},\bar{u}) - \bar{\Gamma}^{\,i}_{jk}(\bar{x},\bar{u}) = 0. \tag{3.4}$$

Conversely, as is easily seen, if this condition is satisfied, then the point transformation (3.1) defines an affine collineation.

From the above discussion, when an infinitesimal point transformation (1.1) defines an infinitesimal affine collineation, we have, from (3.4),

$$X\Gamma^{\,i}_{jk} = \xi^i_{;j;k} + R^i_{jkl}\,\xi^l - \Gamma^i_{jk}/^a\,\xi^b_{;a}u_b = 0. \tag{3.5}$$

Conversely, if this condition is satisfied, the tensor equations (3.5) become $\partial\Gamma^i_{jk}/\partial x^1 = 0$ in a suitable coordinate system such that $\xi^i = \delta^i_1$, and consequently $\Gamma^i_{jk}(\bar{x}.\,\bar{u}) = \Gamma^i_{jk}(x,u)$. On the other hand, from the transformation law of Γ^i_{jk}, we have $\bar{\Gamma}^i_{jk}(\bar{x},\bar{u}) = \Gamma^i_{jk}(x,u)$. Thus, the equations (3.4) hold. We have

Theorem 3.1. A necessary and sufficient condition that an infinitesimal transformation (1.1) be an infinitesimal affine collineation is that the Lie-derivatives of the components of the affine connection vanish.

From (1.11), we have

Theorem 3.2. In order that an infinitesimal transformation (1.1) be an infinitesimal affine collineation, it is necessary and sufficient that the Lie-operation and the covariant operation be interchangeable.

As is seen from the proof of theorem 3.1, when the space admits an infinitesimal affine collineation, there exists a coordinate system such that $\partial\Gamma^i_{jk}/\partial x^1 = 0$. If this condition is satisfied, the finite equations $\bar{x}^i = x^i + \delta^i_1\,t$ satisfy (3.4), in other words, the space admits a one-parameter group of affine collineations generated by the infinitesimal affine collineation $\bar{x}^i = x^i + \delta^i_1\,dt$.

If we take a coordinate system for which $\xi^i = x^i$, the equations (3.5) become $\Gamma^i_{jk,a}x^a + \Gamma^i_{jk} = 0$. From this condition, we can see that our space admits a one-parameter group of affine collineations $\bar{x}^i = x^i\,t$ generated by $\bar{x}^i = x^i + x^i dt$. Thus we have

Theorem 3.3. If the space admits an infinitesimal affine collineation, the space

admits also a one-parameter group of affine collineations generated by the infinitesimal affine collineation.

Theorem 3.4. A necessary aud sufficient condition that the space admit a one-parameter group of affine collineations is that there exist a coordinate system in which $\partial\Gamma^i_{jk}/\partial x^1=0$ *or* $\Gamma^i_{jk,a}\,x^a+\Gamma^i_{jk}=0$.

The following theorem can also be easily proved:

Theorem 3.5. If $X_\mu f$ *are the symbols of* r *infinitesimal affine collineations, then* $C^\lambda X_\lambda f$ *is also that of an infinitesimal affine collineation, where* C^λ *are arbitrary constants not all zero.*

From theorem 2.1 and theorem 3.3, we have

Theorem 3.6. If $\xi^i_\mu(x)$ *are the vectors defining* r *one-parameter groups of affine collineations, then so are* $X_\mu\xi^i_\nu$.

When $X_\mu f$ are the generators of an r-parameter group G_r, the transformations of this group consist of the transformations of one-parameter group generated by the infinitesimal transformation with the symbol $C^\lambda X_\lambda f$ (C^λ: constants) and of the products of such transformations. Therefore, from theorem 3.5, we have

Theorem 3.7. If each of r *generators of an* r*-parameter group* G_r *is a generator of a one-parameter group of affine collineations, every transformation of* G_r *is also an affine collineations.*

From Lie's second fundamental theorem and theorem 3.7, we have

Theorem 3.8. If $X_\mu f$ *are generators of a complete set of one-parameter groups of affine collineations, they are generators of an* r*-parameter group of affine collineations.*

4. We shall find the integrability conditions of the equations

$$X\Gamma^i_{jk}=\xi^i_{;j;k}+R^i_{jkl}\,\xi^l-\Gamma^i_{jk}{}/^a\,\xi^b_{;a}u_b=0. \tag{4.1}$$

(4.1) are written in the standard form:

$$\xi^i_{;j}=\eta^i_j,\qquad \xi^i{}/^k=0,\qquad \eta^i_{j;k}=-R^i_{jkl}\,\xi^l+\Gamma^i_{jk}{}/^a\eta^b_a u_b,\qquad \eta^i_j{}/^k=\xi^a\,\Gamma^i_{ja}{}/^k \tag{4.2}$$

with $N(N+1)$ unknown functions ξ^i and η^i_j, independent variables being x^i and u_j; the integrability conditions can be obtained in the usual way. But, using the formulae already obtained, we can find these conditions as follows:

If we substitute (4.1) into (1.13) and (1.14), then we get the conditions

$$XR^i_{jkl}=0 \tag{4.3}$$

and

$$X\Gamma^i_{jk}{}/^l=0 \tag{4.4}$$

respectively.

Our conditions are (4.3), (4.4) and those obtainable from them by repeated covariant differentiation and partial differentiation with respect to u, For this purpose, we shall use

freely (1.11) and (1.12), that is to say, the fact that, under the condition (4.1), the Lie-operation is interchangeable with both covariant differentiation and partial differentiation with respect to u_j.

We first consider the condition (4.4). By repeated differentiation with respect to u, we have a set of conditions

$$XT^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s}=0 \quad (s=1,2,\cdots\cdots), \tag{4.5}$$

but, from the relations $Xu_j=0$ and the homogeneity properties of the left hand sides of the above equations with respect to u, the final equations of (4.5) include all the preceding ones. Thus, we can rewrite (4.5) as

$$XT^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s}=0. \tag{4.6}$$

Differentiating (4.6) covariantly, we have a set of conditions

$$XT^{i}_{jk}/_{l_1}/_{l_2}\cdots\cdots/_{l_s};_{m_1};_{m_2};\cdots\cdots;_{m_t}=0 \quad (t=1,2,\cdots\cdots). \tag{4.7}$$

We consider the partial derivatives of (4.7) for $t=1$ with respect to u_n. If we apply the formula (1.9) to the tensor $T^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s}$ and next apply the Lie-operation to the resulting equations, we can see that

$$(XT^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s};_{m_1})/^{n}=XT^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s};_{m_1}/^{n}$$

are expressible in terms of $XT^{i}_{jk}/^{n}$, $XT^{i}_{jk}/_{l_1}/_{l_2}/\cdots\cdots/_{l_s}$ and $XT^{i}_{jk}/_{l_1}/_{l_2}\cdots\cdots/_{l_s}/^{n};_{m_1}$ which is the same type as the left hand side of (4.7) for $t=1$. Thus, the partial derivatives of (4.7) for $t=1$ with respect to u_n give no new conditions. This argument can be repeated for $t=2,3,\cdots\cdots$.

Next, we consider (4.3). The equations $XR^{i}_{jkl}/^{m}=0$ obtained from (4.3) by the partial differentiation with respect to u_m give no new conditions. For, if we apply the Lie-operation to both sides of the identities

$$R^{i}_{jkl}/^{m}=T^{i}_{jk}/^{m};_{l}-T^{i}_{jl}/^{m};_{k},$$

then we can see that $(XR^{i}_{jkl})/^{m}=XR^{i}_{jkl}/^{m}$ are the consequences of (4.7).

By repeated covariant derivation, we have a set of new conditions

$$XR^{i}_{jkl};_{m_1};_{m_2};\cdots\cdots;_{m_t}=0 \quad (t=1,2,\cdots\cdots). \tag{4.8}$$

By the similar method as in the precedent case, the partial derivatives of (4.8) for $t=1$ with respect to u_n give no new conditions. For, if we apply the formula (1.9) to the tensor R^{i}_{jkl} and next apply the Lie-operation to the resulting expressions, it can be seen that $(XR^{i}_{jkl};_{m_1})/^{n}=XR^{i}_{jkl};_{m_1}/^{n}$ are expressible in terms of $XT^{i}_{jk}/^{n}$, XR^{i}_{jkl} and $XR^{i}_{jkl}/^{n};_{m_1}$, but the expressions $RX^{i}_{jkl}/^{n};_{m_1}=0$ are the consequences of (4.7) for t=2.

Thus, our integrability conditions are that a set of equations (4.3), (4.6), (4.7) and (4.8) is algebraically consistent in ξ^{i} and $\xi^{i}_{;j}$. We shall remark here that the solutions ξ^{i} of (4.1) or (4.2) do not contain the variables u_i.

Theorem 4.1. A necessary and sufficient condition that the space of hyperplanes admit an infinitesimal affine collineation is that

$$(4.9)\qquad \begin{cases} XI'^{i}_{jk/l_1/l_2/\cdots\cdots/l_s;m_1;m_2;\cdots\cdots;m_p} = 0 & (p=0,1,2,\cdots\cdots) \\ XR^{i}_{jkl;m_1;m_2;\cdots\cdots;m_t} = 0 & (t=0,1,2.\cdots\cdots) \end{cases}$$

be algebraically consistent in ξ^i and $\xi^i_{;j}$.

The left hand sides of the integrability conditions are linear homogeneous in ξ^i and $\xi^i_{;j}$, so that we have

Theorem 4.2. A necessary and sufficient condition that the space admit r linearly independent infinitesimal affine collineations is that there exist $N(N+1)-r$ independent equations in (4.9).

Furthermore, an arbitrary solution of (4.1) is a linear combination (with constant coefficients) of such r vectors $\xi^i_\lambda(x)$. From theorems 2.1 and 3.1, $X_\mu\xi^i_\nu$ are solutions of (4.1), so that they are linear combineations of $\xi^i_\nu(x)$ with constant coefficients. Thus, from theorem 3.8, we have

Theorem 4.3. A necessary and sufficient condition that the space admit an r-parameter group of affine collineations is that there exist $N(N+1)-r$ independent solutions in (4.9).

If the equations (4.1) are completely integrable, then $XI'^{i}_{jk/l}=0$ and $XR^{i}_{jkl}=0$ must be identities in ξ_i and $\xi^i_{;j}$, and consequently $I'^{i}_{jk/l}=0$ and $R^{i}_{jkl}=0$. Thus, we have

Theorem 4.4. In order that the space admit a group of infinitesimal affine collineations of maximum order, it is necessary and sufficient that the space be affinely flat.

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NON EUCLIDEAN GEOMETRY IN FINSLER SPACES

Yasuo NASU

(Received January 10, 1952)

On a projectively connected space whose group of holonomy fixes a non-degenerate hyperquadric, some interesting results have been obtained by S. Sasaki and K. Yano [1]. On the other hand, T. Otsuki has reached to the same results. by a slightly different way [2].

This paper deals with the consideration for a general space with projective connextion.

1. In a projectively connected space, we use a *repère semi-naturel* $[R_o, R_j]$. The projective connexion is defined by

$$dR_o = (\Phi_k dx^k + \Psi_k dp^k)R_o + R_k dx^k,$$

$$dR_j = (\gamma^o_{jk} dx^k + D^o_{jk} dp^k)R_o + (\gamma^i_{jk} dx^k + D^i_{jk} dp^k)R_i . \tag{1.1}$$

Then the coefficients of connexion are usually defined by the following formulae:

$$\Gamma^o_{jk} = \gamma^o_{jk}, \qquad \Gamma^i_{jk} = \gamma^i_{jk} - \delta^i_j \Phi_k,$$

$$\Gamma^i_{ok} = \gamma^i_{ok} = \delta^i_k,$$

$$C^o_{jk} = D^o_{jk}, \qquad C^i_{jk} = D^i_{jk} - \delta^i_j \Psi_k,$$

$$C^i_{ok} = D^i_{ok} = 0.$$

Generally the coefficients of connexion are the functions of the x's. and the p's, in which (x^i, p^i) and $(x^i, \rho p^i)$ $(\rho \neq 0)$ are being the same element of the manifold. Accordingly, as the connextion is independent of ρ, we can see that Φ_k, γ^o_{jk}, γ^i_{jk}, together with Γ^o_{jk} and Γ^i_{jk} are the functions of degree zero in the p's, and that Ψ_k, D^o_{jk}, D^i_{jk}, together with C^o_{jk} and C^i_{jk} are the functions of degree -1 in the p's. A coordinate transformation is given by

$$\bar{x}^i = \bar{x}^i(x^1, \cdots\cdots, x^n) \qquad (i, j, k \cdots\cdots = 1, 2, \cdots\cdots, n), \tag{1.2}$$

where $\bar{x}^i$ are analytical functions of the x's and the functional determinant is different from zero. The coefficients of connexion Γ^o_{jk}, Γ^i_{jk}, C^o_{jk}, C^i_{jk} are transformed by (1.2) into

$$\bar{\Gamma}^o_{jk} = \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k}\Gamma^o_{ab} + \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial^2 x^b}{\partial \bar{x}^k \partial \bar{x}^c}\frac{\partial \bar{x}^c}{\partial x^d} p^\alpha C^o_{ab},$$

$$\bar{C}^o_{jk} = \frac{\partial x^a}{\partial \bar{x}^j}\frac{\partial x^b}{\partial \bar{x}^k} C^o_{ab}, \tag{1.3}$$

$$\bar{\Gamma}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^a}\left(\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Gamma^a_{bc} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k} + \frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial^2 x^c}{\partial \bar{x}^k \partial \bar{x}^d}\frac{\partial \bar{x}^d}{\partial x^e} p^e C^a_{bc}\right),$$

$$\bar{C}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k} C^a_{bc.}$$

Furthermore, we must consider a transformation of the *repère semi-naturel* $[R_o, R_j]$, namely:

$$R_o = R_o \ , \qquad R_j = R_j + \lambda_j R_o \quad (\lambda_j \text{ is a vector}), \tag{1.4}$$

which is usually called a transformation of the hyperplane at infinity. Thne Γ^o_{jk}, Γ^i_{jk}, C^o_{jk} C^i_{jk} are transformed by (1.4) into

$$\begin{aligned}
\bar{\Gamma}^o_{jk} &= \Gamma^o_{jk} + \frac{\partial \lambda_j}{\partial x^k} - \lambda_i \Gamma^i_{jk} - \lambda_j \lambda_k, \\
\bar{C}^o_{jk} &= C^o_{jk} + \frac{\partial \lambda_j}{\partial p^k} - \lambda_i C^i_{jk}, \\
\bar{\Gamma}^i_{jk} &= \Gamma^i_{jk} + \delta^i_j \lambda_k + \delta^i_k \lambda_j, \\
\bar{C}^i_{jk} &= C^i_{jk}.
\end{aligned} \tag{1.5}$$

The proofs of (1.3). and (1.5) are given by K. Yano [3].

In the general space with the projective connexion (1.1), if the group of holonomy fixes na non degenerate hyperquadric at a tangential point x^i with the equation

$$Q_{n-1}: \qquad a_{\lambda\mu} X^\mu X^\lambda = 0 \qquad (det|a_{\lambda\mu}| \neq 0, \ a_{\lambda\mu} = a_{\lambda\mu}) \tag{1.6}$$

where $X_{\mu\lambda}$ being a projective tensor and X_λ a projective vector, then we must have the condition

$$D(a_{\lambda\mu} X^\lambda X^\mu) = \tau a_{\lambda\mu} X^\lambda X^\mu,$$

where we denote by D the covariant differential and by τ a scalar factor. Since $DX_\lambda = 0$ for an arbitrary projective vector we have the following equations:

$$(Da_{\lambda\mu}) X^\lambda X^\mu = \tau \ a_{\lambda\mu} X^\lambda X^\mu.$$

Therefore , a necessary and sufficient condition in order that the group of holonomy fixes the hyperquadric Q_{n-1} is as follows:

$$Da_{\lambda\mu} = \tau a_{\lambda\mu} \quad (\tau = \tau_k dx^k + \tau'_k dp^k). \tag{1.7}$$

This is also expressed in the following form:

$$\begin{aligned}
\frac{\partial a_{\lambda\mu}}{\partial x^k} - \gamma^\nu_{\mu k} a_{\lambda\nu} - \gamma^\nu_{\lambda k} a_{\mu\nu} &= \tau_k a_{\lambda\mu}, \\
\frac{\partial a_{\lambda\mu}}{\partial p^k} - D^\nu_{\mu k} a_{\nu\lambda} - D^\nu_{\lambda k} a_{\mu\nu} &= \tau'_k a_{\lambda\mu},
\end{aligned} \tag{1.8}$$

When $a_{oo} \neq 0$, we can put, without loss of generality,

$$a_{oo} = \varepsilon \ (= \pm 1). \tag{1.9}$$

Hence putting $\lambda=0$, $\mu=0$ in (1.8) we get the following relations

$$\begin{aligned} &\tau_k + 2\,\Phi_k = -2\,\varepsilon\,a_k, \quad (a_k = a_{ok} = a_{ko}), \\ &\tau'_k = -2\,\Psi_k. \end{aligned} \tag{1.10}$$

Furthermore, putting $\lambda=0$, $\mu=i$ in (1.8), we get the following relations:

$$\frac{\partial a_j}{\partial x^k} - \Phi_k a_j - \gamma^o_{jk}\,\varepsilon - a_{jk} - \gamma^l_{jk}\,a_l = \tau'_k a_j,$$

$$\frac{\partial a_j}{\partial p^k} - \Psi_k\,a_j - C^o_{jk}\,\varepsilon - C^l_{jk}\,a_l = \tau'_k a_j.$$

The above relations and (1.9) show us that

$$\begin{aligned} &\Gamma^o_{jk} + \frac{\partial}{\partial x^k}(-\varepsilon a_j) - (-\varepsilon a_l)\Gamma^l_{jk} - (-\varepsilon a_j)(-\varepsilon a_k) = -\varepsilon g_{jk}, \\ &C^o_{jk} + \frac{\partial}{\partial p_k}(-\varepsilon a_j) - (-\varepsilon a_l)\Gamma^l_{jk} = 0, \end{aligned} \tag{1.11}$$

where $g_{ij} = a_{ij} - \varepsilon a_i a_j$.

Similarly, putting $\lambda=i$, $\mu=j$, we obtain from (1.10) and (1.11) the following results:

$$\begin{aligned} &\frac{\partial g_{ij}}{\partial x_k} - \overline{\Gamma}^l_{ik} g_{jl} - \overline{\Gamma}^l_{jk} g_{il} = 0, \\ &\frac{\partial g_{ij}}{\partial p_k} - \overline{C}^l_{ik}\,g_{jl} - \overline{C}^l_{jk}\,g_{il} = 0. \end{aligned} \tag{1.12}$$

where $\overline{\Gamma}^i_{jk} = \gamma^i_{jk} - \delta^i_j\,\Phi_k + (-\varepsilon a_j)\,\delta^i_k + (-\varepsilon a_k)\,\delta^i_j$ and $\overline{C}^i_{jk} = D^i_{jk} - \delta^i_j\,\Psi_k$. Now we can put $g_{it}\,\overline{C}^t_{jk} = \overline{C}_{jik}$, and assume the followig condition:

(i) *$\overline{C}_{jik}$ are symmetric with respect to i, j, k.*

Then we get from the second equation of (1.12),

$$\frac{\partial g_{ij}}{\partial p^k} = \overline{C}_{ijk},$$

so that the differential equations

$$\frac{\partial^2 \overline{F}}{\partial p^i\,\partial p^j} = g_{ij} \tag{1.13}$$

are completely integrable by the reason of (i)

If a solution of (1.13) satisfies the following conditions

(ii) *$\overline{F}$ $(x^i, p^i) > 0$ or < 0 for every $p^1, p^2, \cdots\cdots, p^n$ not all zero,*

(iii) *$\overline{F}$ is a homogeneous function of degree 2 in the p's,*

then, following E. Cartan [4] we can introduce the connexion of the Finsler space whose fundamental metric function is given by $F = k\sqrt{\overline{F}(x^i, p^i)}$ (k being a constant) and the

coefficients of connexion are given by the following formula:

$$\overline{\Gamma}^{i}_{jk} = \left\{ {i \atop jk} \right\} - g^{im} \left(\overline{C}_{jkr} \frac{\partial G^r}{\partial p^m} - \overline{C}_{mjr} \frac{\partial G^r}{\partial p^k} \right) ,$$

where $\left\{ {i \atop jk} \right\}$ is the Christoffel symbol and G^r are given by

$$G_h = g_{hr} G^r = \frac{1}{2} \left(\frac{\partial^2 F}{\partial p^h \partial x^k} p^k - \frac{\partial F}{\partial x^h} \right).$$

By these arguments, we have the following

Theorem. In a general space with projective connexion, when the group of holonomy fixes any non-degenerate hyperquadric, the coefficients of the projective connexion are given by those of a Finsler space.

2. In this paragraph, following S. Sasaki [5], we show that a metric can be defined for a general space with projective connexion whose group of holonomy fixes a hyperquadric $a_{\mu\lambda} X_\lambda X_\mu = 0$.

We consider an arbitrary continuous curve $x^i = x^i(t)$. The arc-length ds of this curve is defined by

$$\cos \frac{ds}{k} = \frac{1}{\sqrt{a_{\lambda\mu} u^\mu u^\lambda}} , \qquad (k \neq 0)$$

where k is an arbitrary constant and u_λ are defined as follows:

$$u^0 = \varepsilon \left(1 - a_{0j} \left(x, \frac{dx}{dt} \right) dx^j \right), \quad u^k = dx^k,$$

where $dx^k = \frac{dx^k}{dt} dt$.

From this definition of the metric, we can easily find by simple calculation the following result:

$$\cos \frac{ds}{k} = \frac{1}{\sqrt{\varepsilon + g_{ij} \left(x, \frac{dx}{dt} \right) dx^i \, dx^j}} . \tag{2.1}$$

If $g_{ij} \left(x, \frac{dx}{dt} \right)$ is positive definite, we can put $\varepsilon = 1$ and get, by expanding (2.1) in power series,

$$ds^2 = k^2 g_{ij} \left(x, \frac{dx}{dt} \right) dx^i \, dx^j .$$

If $g_{ij} \left(x, \frac{dx}{dt} \right)$ is negative definite, then by putting $\varepsilon = -1$, the imaginary distance is defined by (2.1).

3. We have discussed the projective connexion of the general space whose group of holonomy fixes the hyperquadric Q_{n-1}. It is obvious that the Q_{n-1} is expressible by

$$\varepsilon \left(X^0 \right)^2 + g_{ij} \, X^i \, X^j = 0.$$

The projective connexion is then expressed by the following formulae:

$$dR_0 = R_i\, dx^i\,,$$

$$dR_j = g_{jk} dx^k R_0 + (\overline{\Gamma}^i_{jk} dx^k + \overline{C}^i_{jk} dp^k) R_i.$$

S. Sasaki formerly studied the relation between the metric of a projectively (or conformally) connected space whose group of holonomy fixes a non-degenerate hyperquadric and the non-euclidean geometry with a hyperquadric as the absolute figure [6].

We can prove that there is the same relation, in other words, the Klein's representation is applicable to the hyperquadric (as the absolute figure) and to the metric which we have introduced in this paper.

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A THEOREM ON FIXED POINTS AND ITS APPLICATION TO THE THEORY OF DIFFERENTIAL EQUATIONS

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1. Let B be a Banach space, and K a convex compact subset of B. It is well known (Cf. [1]*, [3], [5]) that any absolutely continuous mapping that maps K into itself has at least one fixed point in K. The present paper deals with the situation of dependency of fixed points on a parameter σ which, being a variable element of a second Banach space B_σ, appears in the relation of the mapping, and also one of the applications to the problem of dependency (Abhängigkeit) in the theory of ordianary differential equations.

Now, let $y=\varphi(x,\sigma)$, where $x\in B$ and $\sigma\in B_\sigma$, be an absolutely continuous mapping which maps K into itself, and let x_σ denote fixed points of this mapping corresponding to σ, and $\mathfrak{M}_\sigma$, the set of all x_σ for each $\sigma\in B_\sigma$. Thus our first purpose is to prove the

Theorem. Suppose $\varphi(x,\sigma)$ continuous in (x,σ) at the point $(x,\sigma_0)\in B\times B_\sigma$ for each $x\in K$ and a fixed $\sigma_0\in B_\sigma$, and let $\rho(x,\mathfrak{M})$ denote the distance between a point x and a subset $\mathfrak{M}$. Then we have

$$\lim_{\sigma\to\sigma_0}\sup_{x_\sigma\in\mathfrak{M}_{\sigma_0}}\rho(x_\sigma,\mathfrak{M}_\sigma)=0,$$

that is, the subset $\mathfrak{M}_\sigma$ can be contained in an arbitrarily near neighbourhood of $\mathfrak{M}_{\sigma_0}$, whenever σ is sufficiently near σ_0.

A direct consequence of this theorem is the following

Corollary. Under the same assumption of the theorem, if the subset $\mathfrak{M}_{\sigma_0}$ consists of a single point x_{σ_0},

$$\lim_{\sigma\to\sigma_0}x_\sigma=x_{\sigma_0}$$

for any selection of x_σ from $\mathfrak{M}_\sigma$.

2. **Proof of the theorem.** For the proof of the theorem we have to prove previously the following

Lemma. Under the same assumption of the theorem, if $\{\sigma_n\}$ be a sequence of points of B_σ, converging to a point $\sigma_0\in B_\sigma$, then, for every sequence $\{x_{\sigma_n}\}$ of fixed points corresponding to $\{\sigma_n\}$, there exists a subsequence $\{x_{\sigma_{n'}}\}$, such as converges to a point

$$x_{\sigma_0}\in\mathfrak{M}_{\sigma_0}.$$

Proof. Since $\{\sigma_n\}$ converges to σ_0, $\{\varphi(x,\sigma_n)\}$ converges to $\varphi(x,\sigma)$, and that uniformly for $x\in K$, because of the compactness of K. Let $\{\varepsilon_n\}$ be a sequence of possitive numbers converging to 0, and then there exists such a sequence $\{N_n\}$ of positive integers depending on ε_n, but not on x, as

* The numbers in the brackets refer to the references at the end of the paper

$$\| \varphi(x_\sigma, \sigma_i) - \varphi(x_{\sigma_i}, \sigma_0) \| < \varepsilon_n \quad \text{for} \quad i \geqq N_n,$$

which is equivalent to

$$\| x_{\sigma_i} - \varphi(x_{\sigma_i}, \sigma_0) \| < \varepsilon_n \quad \text{for} \quad i \geqq N_n,$$

since x_{σ_i} is a fixed point corresponding to σ_i. Therefore there exists such a subsequence $\{x_{\sigma_{n'}}\}$ of $\{x_{\sigma_n}\}$, as

$$\| x_{\sigma_{n'}} - \varphi(x_{\sigma_{n'}}, \sigma_0) \| < \varepsilon_{n'}.$$

Since $\{x_{\sigma_{n'}}\}$ is a subset of the compact set K, it has a subsequence $\{x_{\sigma_{n''}}\}$ converging to a point $x_0 \in K$, and, therefore, owing to the continuity of φ, we have

$$\lim_{n'' \to \infty} \varphi(x_{\sigma_{n''}}, \sigma_0) = \varphi(x_0, \sigma_0).$$

On the other hand

$$\| x_{\sigma_{n''}} - \varphi(x_{\sigma_{n''}}, \sigma_0) \| < \varepsilon_{n''}$$

and since ε_n converges to 0,

$$x_0 = \lim_{n'' \to \infty} \varphi(x_{\sigma_{n''}}, \sigma_0) = \varphi(x_0, \sigma_0).$$

Hence we have $x_0 \in \mathfrak{M}_{\sigma_0}$, which proves the lemma.

Proof of the theorem. Suppose the conclusion of the theorem is false. Then there exist a subsequence $\{\sigma_{n'}\}$ of the sequence $\{\sigma_n\}$ and a positive constant c, such as

$$\sup_{x_{\sigma_{n'}} \in \mathfrak{M}_{\sigma_{n'}}} \rho(x_{\sigma_{n'}}, \mathfrak{M}_{\sigma_0}) > c.$$

Therefore, there exists such a corresponding sequence $\{x_{\sigma_{n'}}\}$, as

$$\rho(x_{\sigma_{n'}}, \mathfrak{M}_{\sigma_0}) > c,$$

and then

$$\rho(x_{\sigma_{n'}}, x_{\sigma_0}) > c \quad \text{for all} \quad x_{\sigma_0} \in \mathfrak{M}_{\sigma_0},$$

which contradicts to the lemma.

3. **Application.** Let R^1 denote the space of real numbers, t its element, I an interval: $t^0 \leqq t \leqq t^1$, R^n an n-dimensional vector space, and x its element normed by

$$\| x \| = \sum_{i=1}^{n} | x_i | \quad \text{where} \quad x = (x_1, \cdots\cdots, x_n).$$

Moreover, let $f(t)$ denote a vector function, components being

$$f_1(t), \cdots\cdots, f_n(t).$$

The totality of vector functions $f(t)$, continuous in the interval I, forms, as is well-known, a function space C^n, which is a Banach space. Let $f(t)$ be denoted by $\boldsymbol{f}$, as element of C^n, and the corresponding norm, by $\| \boldsymbol{f} \|_{C^n}$ defined by

$$\| \boldsymbol{f} \|_{C^n} = \max_{t \in I} \| f(t) \|.$$

Now we consider a differental equation:

$$\frac{dx(t)}{dt} = f(x(t), t) \tag{1}$$

in the vector space R^n, $f(x,t)$ continuous in x and t, under the initial condition that when $t=\tau \in I$, $x=\xi$. As is well known, the problem of the equation (1) can be transformed into that of the integral equation:

$$x(t)=\xi+\int_\tau^t f(x(t),t)\,dt. \tag{2}$$

If we put

$$\xi+\int_\tau^t f(x(t),t)\,dt=\varphi(x(t),f,\tau,\xi)=y(t), \tag{3}$$

we have a mapping $y=\varphi(x,\ f,\ \tau,\ \xi)$ of $C^n\times C^{n+1}\times R^1\times R^n$ into C^n.

The following equality and inequalities:

$$\|\varphi(x(t),\ f,\ \tau,\ \xi)-\varphi(x(t),\ f,\ \tau,\ \tilde{\xi})\|=\|\xi-\tilde{\xi}\|, \tag{4}$$

$$\|\varphi(x(t),\ f,\ \tau,\ \xi)-\varphi(x(t),\ f,\ \tilde{\tau},\ \xi)\|\leqq\|f\|_{C^{n+1}}\cdot|\tau-\tilde{\tau}|, \tag{5}$$

$$\|\varphi(x(t),\ f,\ \tau,\ \xi)-\varphi(x(t),\ \tilde{f},\ \tau,\ \xi)\|\leqq\|f-\tilde{f}\|_{C^{n+1}}\cdot|t^1-t^0| \tag{6}$$

give the equality and inequalities respectively:

$$\|\varphi(x,\ f,\ \tau,\ \xi)-\varphi(x,\ f,\ \tau,\ \tilde{\xi})\|_{C^n}=\|\xi-\tilde{\xi}\|, \tag{7}$$

$$\|\varphi(x,\ f,\ \tau,\ \xi)-\varphi(x,\ f,\ \tilde{\tau},\ \xi)\|_{C^n}\leqq\|f\|_{C^{n+1}}\cdot|\tau-\tilde{\tau}|, \tag{8}$$

$$\|\varphi(x,\ f,\ \tau,\ \xi)-\varphi(x,\ \tilde{f},\ \tau,\ \xi)\|_{C^n}\leqq\|f-\tilde{f}\|_{C^{n+1}}\cdot|t^1-t^0|. \tag{9}$$

In the next place, we consider the product space $C^{n+1}\times R^1\times R^n$, and denote its element by $\sigma=(f,\ \tau,\ \xi)$, and the function $\varphi(x,\ f,\ \tau,\ \xi)$ by $\varphi(x,\ \sigma)$.

We restrict the domain of f to $\mathfrak{D}_f$, where the norm $\|f\|_{C^{n+1}}$ is bounded by a positive number M. The equality (7) and inequalities (8), (9) give the inequality:

$$\begin{aligned}\|\varphi(x,\ \sigma)-\varphi(x,\ \tilde{\sigma})\|_{C^n}&\leqq\|\xi-\tilde{\xi}\|+\|f\|_{C^{n+1}}\cdot|\tau-\tilde{\tau}|\\&\quad+|t^1-t^0|\cdot\|f-\tilde{f}\|_{C^{n+1}}\end{aligned} \tag{10}$$

where $\sigma=(f,\ \tau,\ \xi)$, $\tilde{\sigma}=(\tilde{f},\ \tilde{\tau},\ \tilde{\xi})$. This inequality shows that the mapping $\varphi(x,\ \sigma)$ is continuous in σ, and so also uniformly whenever $f\in\mathfrak{D}_f$, and $\tau\in I$, or what is the same, $\sigma\in\mathfrak{D}_f\times I\times R^n$. That $\varphi(x,\sigma)$ is continuous in x being easily verified, we can conclude that $\varphi(x,\sigma)$ is continuous in $(x,\ \sigma)$. Moreover the function $y(t)=\varphi(x(t)\sigma)$ is, as is easily shown, equicontinuous in t, and therefore the mapping $y=\varphi(x,\ \sigma)$ is compact ([4], p. 51, Th. 3. 1). Hence the mapping φ is absolutely continuous.

Thus, as is well known, the existence of solutions of the integral equation (2) and accordingly of the differential equation (1) is verified as fixed points of the mapping $\varphi(x,\sigma)$. Moreover our theorem can be transformed into the following theorem of the theory of differential equations:

A solution of the differential equation (1), where $\sigma=(f,\ \tau,\ \xi)$, can be contained in an arbitrarily near neighbourhood of the set of the solutions $x(t;\ \sigma_0)$, where $\sigma_0=(f_0,\ \tau_0,\ \xi_0)$, whenever σ is sufficiently near σ_0, or in other words, every solution $x(t;\sigma_0)$ can be

uniformly and arbitrarily approximated to some solution $x(t;\sigma_0)$ *as* $\boldsymbol{f}$ *approches uniformily and sufficiently to* $\boldsymbol{f}_0$, τ *and* ξ *sufficiently to* τ_0 *and* ξ_0 *respectively.*

Our corollary generalizes, as a special case of this theorem, ordinary theorems about the dependency (Abhängigkeit) of ordinary differential equations (Cf. [2], pp. 81—88, and pp. 142—151), and it is moreover to be noticed that the theorem just mentioned does not require the condition of uniqueness of the solution $x(t;\sigma_0)$.

Remark. A generalization of our propositions (and also existence-theorems concerned with them) to the case where the function $x(t)$ is that on R^1 to a Banach space, can hardly be expected to be verified by means of our procedure, since this procedure is based on the compactness of the mapping $\varphi(x, \cdot)$ (in x), or, what is the same, on Alzelà's theorem (Cf. [4] and [5]), the validity of which is not verified in the general case above mentioned.

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ON A THEOREM OF GELFAND AND NEUMARK AND THE B*-ALGEBRA

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1. Introduction and Preliminaries. In this paper we shall state on a new proof of Gelfand-Neumark's theorem and certain related results in the theory of normed rings.

A Banach algebra (over the field of complex numbers) is called a B*-algebra, when there is defined an involutorial anti-automorphism $x \to x^*$ such that $(\lambda x)^* = \bar{\lambda} x^*$, and $\| x^* x \| = \| x \|^2$. An element x of a B*-algebra R is said hermitean if $x^* = x$, and skew-hermitean if $x^* = -x$. Every element x is uniquely represented as the sum $x = u + iv$, by two hermitean elements u and v.

The spectrum of an element x of a B*-algebra R with the unit element e is the set of all the complex numbers for which $x - \lambda e$ does not have the inverse. The spectrum of any hermitean element consists of real numbers. If the spectrum of an hermitean element x consists only of non-negative (or positive) numbers, x is said non-negative (or positive). In this paper, we will consider only those B*-algebras which has the unit element.

The (real) linear space E of all hermitean elements of R is a Banach space with the original norm. We denote by D and D_c the sets of all hermitean, non-negative and positive elements of R. Gelfand and Neumark [1] (also, [3]) have shown, under the additional assumption that *every* $x^*x + e$ *have the inverse* (in this case the B*-algebra being called a C*-algebra), that D and D_0 is convex. We shall show, in the following, this latter fact *without using the Gelfand-Neumark's assumption, i.e.*, that it holds for every B*-algebra with the unit element.

A (complex-valued) linear functional $f(x)$ on the B*-algebra R is said positive if $f(x^*x) \geqq 0$, for every $x \in R$; we denote the totality of such functionals by Ψ. On the other hand, we put P the totality of (real-valued) linear functionals on the Banach space E which satisfies $f(D) \geqq 0$. It can be easily seen that every $f \in P$ satisfies $f(D_c) > 0$. For every linear closed subspace H in E such that $H \frown D_0$ is empty, there is an $f \in P$ with $f(H) = 0$. This is a consequence of the open convexity of D_0 and the Ascoli-Mazur's theorem.

It is obvious that every functional $f \in \Psi$ define a functional of P when it is restricted on the space E; we shall denote this fact by $\Psi \prec P$; on the other hand, if every functional of P can be extended to the functional of Ψ in the sense indicated in Lemma 8 (next §), we shall denote it by $P \prec \Psi$. It is not obvious in general that $\Psi \neq$ empty and $\Psi \succ P$ for an B*-algebra with the unit element. We shall find an necessary and sufficient condition for these requirements.

The following structure theorem for commutative B*-algebra is originally due to Gelfand and Neumark: *If A is any commutative B*-algebra with the unit element, then A is iso*

morphic and isometric to the algebra $C(\mathfrak{M})$ *of all complex-valued, continuous functions on the compact Hausdorff space of all maximal ideals of* A, *such that* $\| x \| = \sup_{M \in \mathfrak{M}} | x(M) |$ *and* $x^*(M) \cdot x(M)$ *for every* $M \in \mathfrak{M}$. It follows at once that any hermitean element $u \in D_0$ (*or* $\in D$) may be represented as an everywhere positive (or non-negative) -valued continuous function on $\mathfrak{M}$, where $\mathfrak{M}$ denotes the compact space of all maximal ideals of any closed commutative self-adjoint B*-sub-algebra with the unit element e which contains u; any $u \in D$ belongs to D_0 if and only if u is regular. Also, we see that any hermitean element w is the difference of two non-negetive, hermitean elements, that is, $w = w_+ - w_-$, where $w_+ = \frac{1}{2}(|w| + w)$, $w_- = \frac{1}{2}(|w| - w)$ and $w = \sqrt{w^2}$; $|w|, w, w_+, w_-$ commute each other and $w_- \cdot w_+ = 0$. Every $u \in D$ can be put in the form $u = u_1^2$, where $u_1 \in E$, and if $u \in D_0$, u_1 must be regular.

We state the following

Theorem 1. Let R *be any* B*-*algebra with the unit element* e. *Then the following postulates for* R *are equivalent:*

(I). *For every* $x, y \in R$, $x^*x + y^*y = 0$ *implies* $x = y = 0$.

(II). *For every* $x, \in R$, $x^*x + e$ *has the inverse element.*

Proof. We will prove the implication (I)→(II). Since $x^*x = u_+^2 - u_-^2$, $u_+ \cdot u_- = 0$ and $u_+, u_- \in E$, $0 = (xu_-)^*(xu_-) + (u_-^2)^2$, which implies $xu_- = 0$, and $u_-^2 = 0$, so that $x^*x = u_+^2$ and $x^*x + e = e + u_+^2$ is regular. The proof of (II)→(I) is analogons.

Theorem 2. The class of all two-sided ideals J *in any* B*-*algebra* R *with the unit element satisfying the condition*

(1) $x^*x + y^*y \in J$ *implies* $x, y \in J$

is identical with the class of all two-sided ideals J *satisfying the condition*

(2) *For every* $z \in J$, *there is a* $f(x) \in \mathfrak{P}$ *such that* $f(z^*z) > 0$ *and* $f(x^*x) = 0$, *for every* $x \in J$.

Proof. We shall only notice that the residue class algebra of any B*-algebra with the unit element modulo any two-sided ideal is again a B*-algebra, and that every two-sided ideal of a B*-algebra is self-adjoint.

2. The Positiveness Concept in the B*-algedra. We shall prove the additivity of positiveness in any B*-algebra, *without the assumption that every* $x^*x + e$ *have the inverse.*

Let R be any B*-algebra with the unit element e. Let E, D, D_0 be defined as in the above. (*Cf.* [3]).

Lemma 1. If u *and* v *are any element of* D_0, *then* $u + v$ *also belongs to* D_0.

Proof. For the proof, we notice that every $u \in D_0$ can be put in the form $u = u_1 + \delta e$ for some $u_1 \in D_0$ and some $\delta > 0$; that is, $u \in D_0$ if and only if $u + \lambda e$ is regular (to have the inverse) for all $\lambda \geqq -\delta$, for some $\delta > 0$. This can be easily seen by means of the structure theorem for the commutative B*-algebra.

Therefore, it can be easily seen that, in order to prove $u + v \in D_0$, for every u, $v \in D_0$, it suffices to show that, for every $u, v \in D_0$ and for every $\lambda > 0$, $u + v + \lambda e$ be regular.

As it is clear that $u \in D_0$ and $\lambda > 0$ imply $\lambda u \in D_0$, we have only to prove that, for every

u, $v\varepsilon D_0$, $e+u+v$ is regular, also it suffices to show that, for every u, $v\varepsilon D_0$, all $e+\alpha u+(1-\alpha)v$, $0\leqq\alpha\leqq 1$, are regular.

Put $w(\alpha)=e+\alpha u+(1-\alpha)v$; as both u and v, together with all $e+\alpha u$, $e+(1-\alpha)v$ are regular for all α, $0\leqq\alpha\leqq 1$, $w(\alpha)$ is regular if and only if

$\overline{w(\alpha)}=u^{-1}w(\alpha)v^{-1}=(u^{-1}+\alpha e)(v^{-1}+(1-\alpha)e)-\alpha(1-\alpha)e$ is regular.

By the representation theorem for R_u and R_v [1], we have

$$\|(u^{-1}+\alpha e)^{-1}\|=\sup_{M}(\alpha+1/u(M))^{-1}\leqq\|u\|/(\|u\|\alpha+1)$$

and

$$\|(v^{-1}+(1-\alpha)e)^{-1}\|\leqq\|v\|/(\|v\|(1-\alpha)+1);$$

consequently, we have

$$\|(u^{-1}+\alpha e)^{-1}\,\overline{w(\alpha)}\,(v^{-1}+(1-\alpha)e)^{-1}-e\|$$

$$\leqq\alpha(1-\alpha)\|u\|\cdot\|v\|/(1+\|u\|\alpha)(1+\|u\|(1-\alpha))<1,$$

for all α, $0\leqq\alpha\leqq 1$, which proves that $(u^{-1}+\alpha e)^{-1}\overline{w(\alpha)}\,(v^{-1}+(1-\alpha)e)^{-1}$ is regular.

Thus, $w(\alpha)$ is also regular for every α, $0\leqq\alpha\leqq 1$, and for every u and $v\varepsilon D_0$, which completes the proof.

An immediate consequence of the lemma is the following

Lemma 2. If u and v belong to D, then $u+u$ also belongs to D.

Lemma 3. Any $u\varepsilon D$ belongs to D_0 if and only if u is an inner point of D.

Proof. Let $u\varepsilon D_0$ and let $v\varepsilon E$ be any element such that $\|u-v\|<E$, E being determined later on. We shall prove that $v\varepsilon D$.

As $u+\lambda e$ is regular for all real $\lambda\geqq-\lambda_0$, for some $\lambda_0>0$, $v+\lambda e=(u+\lambda e)+(v-u)=(u+\lambda e)(e+w)$, where $\|w\|\leqq\|u-v\|\cdot\|(u+\lambda e)^{-1}\|\leqq M\cdot\varepsilon$, with $M=\sup_{\lambda\geqq-\lambda_0}\|(u+\lambda e)^{-1}\|$. Such M exists by the continuity of $\|(u+\lambda e)^{-1}\|$. It follows, after ε being taken such that $M\cdot\varepsilon<1$, that $(u+\lambda e)(e+w)$ is regular for $\lambda\geqq-\lambda_0$, so that $v+\lambda e$ is positive and regular.

Conversely, if u is any inner point of D, then $u-\lambda e\varepsilon D$ for some $\lambda>0$, so that $u=\lambda e+u_1^2\varepsilon D_0$.

Theorem 3. Let R be any B^*-algebra with the unit element e, and let D_0 and D be the totality of hermitean and positive elements and of hermitean and non-negative elements of R. Then D_0 forms a convex, open cone in the real Banach space E of hermitean elements of R, and D is a convex cone. Moreover, D_0 is the set of all inner points of D.

As this theorem is obtained easily by means of the preceeding lemmas 2, 3, we omit the proof.

Lemma 4. $D_0\frown(-D_0)$ is empty; $D\frown(-D)=(0)$.

Proof. $u+v=0$ for $u, v\varepsilon D_0$ is impossible, because D_0 does not contain 0, by the open-

1) Here R_u may be any commutative self-adjoint subalgebra of R with the unit element e, containing u,

ness of D_0.

(ii) As $u, v \in D$ and $u=-v$ imply that u, v commutes; it follows, by the representation theorem, that $u(M)=-v(M)$, for all $M \in \mathfrak{M}$; now, $u(M) \geqq 0$ and $v(M) \geqq 0$ imply $u=v=0$.

Lemma 5. $x^*x+xx^*=0$ *implies* $x=0$.

Proof. Put $x=u+iv$, where u, v are hermitean, then $x^*x+xx^*=2(u^2+v^2)$, from which we have $u=v=0$ and $x=0$, by the preceeding lemma.

Lemma 6. $x^*x \neq -e$.

Proof. Let $x=u+iv$, $u, v \in E$, and suppose $x^*x=-e$. As $x^*x+xx^*=2(u^2+v^2)$, we have $xx^*=e+2(u^2+v^2)$ and so xx^* is regular ($\in D^0$) (by Lemma 1). On the other hand, $(xx^*)^2=x(x^*x)x^*=-xx^*$, which is impossible, because both xx^* and xx^*+e are regular. This shows that $x^*x \neq -e$.

Lemma 7. If u *is hermitean and* $x^*x=-u^2$ *for some* $x \in R$, *then* u *is not regular.*

Proof. If otherwise, we shall have $(xu^{-1})^*(xu^{-1})=-e$; this is impossible by the reason of the preceeding lemma.

By this lemma, any element x^*x which has the non-positive spectrum cannot belong to $-D_0$, but only to $-(D \frown D_0^C)$. (D_0^C denotes the complement of D_0).

Lemma 8. Let R *be any* B^**-algebra with the unit element* e, *and let* E *be the Banach space of all hermitean elements in* R. *If* $f(u)$ *is any real-valued linear functional, defined on* E, *such that* $f(D) \geqq 0$, *then the functional* $F(x)$ *on* R *defined by*

$F(x)=f(u)+if(v)$, *for* $x=u+iv \in R$, $u, v \in E$,

is a complex-valued linear functional on R *such that* $|F(x)| \leq F(e)\|x\|$;

$F(x^*x) \geqq 0$ *for every* $x \in R$ *if and only if* $f(x^*x) \geqq 0$, *for every* $x \in R$.

Proof. $F(\lambda x)=\lambda F(x)$ and $F(x+z)=F(x)+F(y)$ are obvious. Since $|f(u)| \leq f(e)\|u\|$, $|f(u)|^2 \leq f(e)f(u^2)$ and $2(u^2+v^2)=x^*x+xx^*$, we have $|F(x)|=\{|f(u)|^2+|f(v)|^2\}^{\frac{1}{2}} \leq f(e)^{\frac{1}{2}}(f(u^2+v^2))^{\frac{1}{2}} \leq f(e)\|u^2+v^2\|^{\frac{1}{2}}=f(e)\|\frac{1}{2}(x^*x+xx^*)\|^{\frac{1}{2}} \leq f(e)\|x\|=F(e)\|x\|$.
The last statement is obvious.

Lemma 9. Let G *be a real Banach space, and* H *a linear closed subspace of* G. *Suppose that there is defined a set* K *in* G *such that*

1. $x \in K$ *and* $\lambda > 0$ *imply* $\lambda x \in K$,
2. x *and* $y \in K$ *implies* $x+y \in K$,
3. $x \in K$ *implies* $-x \bar{\in} K$,
4. K *is an open set,*
5. $K \frown H$ *is empty.*

Then the factor space $G-H$ *is a real Banach space in its usual definition of the norm, in which the set* $K-H$ ($\subset G-H$) *satisfies the same postulates* 1.–4.

Proof. Let U, V be the elements of $K-H$, *i.e.*, the cosets $U=u+H$, $V=v+H$ with $u, v \in K$. The validity of 1, 2 and 3 for $K-H$ is obvious. Take any $U \in K-H$, then we can take, by 4, an $\varepsilon > 0$ such that the sphere $\{x / \|u-x\| < \varepsilon, x \in G\}$ is contained in K. Consider the

sphere $\{X/\| U-\times \| < \frac{\varepsilon}{2}\}$ in G^-H. As $\| U-X \| = \inf_{h\varepsilon H} \| u-x+h \|$, there is an $h\varepsilon H$ such that $\| u-x+h \| < \varepsilon$, so that we have $v=x-h\varepsilon K$ and $X=x+H=(v+h)+H=v+H$ and $X\varepsilon K-H$. This proves 4. for $K-H$.

The following theorem is a modification of the Ascoli-Mazur's Theorem.

Theorem 4. Let G, H be a real Banach space and its any linear closed subspace. If there is defined a set K satisfying the following conditions:

1. *$x\varepsilon K$ and $\lambda>0$ imply $\lambda x\varepsilon K$,*
2. *x and $y\varepsilon K$ imply $x+y\ \varepsilon K$,*
3. *$x\varepsilon K$ implies $-x\bar{\varepsilon}K$,*
4. *K is an open set,*
5. *$K\frown H$ is empty.*

Then there exists a linear functional $f(x)$ on G satisfying $f(K)>0$ and $f(H)=0$.

3. The Positive Functional on the B*-algebra.

Theorem 5. Let R, E and D_0 be defined as above and let H be any linear closed subspace of E. suchthat $H\frown D_0$ is empty. Then there is an $f(x)\varepsilon P$ such that $f(H)=0\,(f(D_0)>0)$.

Cor. 1. For every $v\varepsilon D\frown D_0{}^C$, there is a $f(x)\varepsilon P$ with $f(v)=0$.

Cor. 2. For every $v\varepsilon D\frown D_0{}^C$, there is a $f(x)\varepsilon P$ with $f(v)>0$.

Proof. Put $u=e+\alpha v, (-\infty<\alpha<\infty)$, then there is an $\alpha_1<0$ such that $u_1=e+\alpha_1 v\varepsilon D\frown D_0{}^C$. If we put H the linear closed subspace of E spanned by the vector u_1, then $H\frown D_0=$ empty is obvious, so that, we obtain a $f(x)\varepsilon P$ with $f(H)=0$, which is the required one.

Lemma 10. If R satisfies the condition:

(III). *$x^*x+y^*y+\cdots+z^*z\neq -e$, for every finite set of elements $x,y,\ldots,z$,*

then, there is a linear functional $f(x)\varepsilon\mathfrak{P}$, that is, $\mathfrak{P}$ is not empty.

Proof. At first, we consider all those elements w with $w=x^*x=-u$, where $x\varepsilon R$ and $u\varepsilon E$. Let G be the totality of all positive linear combinations of any finite number of such elements. It is obvious that $\lambda G\subset G$ for every $\lambda>0$ and that every element of G is identical with a certain $-u$, $u\varepsilon D$ (Lemma 2), where u cannot belong to D_0. For otherwise, we have a relation

$$x^*x+y^*y+\cdots+z^*z=-u,\ u \text{ being regular,}$$

which imply the relation

$$x_1{}^*x_1+y_1{}^*y_1+\cdots+z_1{}^*z_1=-e,$$

which contradicts to (III).

Thus, any $w\varepsilon G$ does not belong to $-D_0$. Also the linear closed subspace spanned by G does not contain any point of D_0. For, if $\bar{G}=G-G$ does contain an $u\varepsilon D_0$, then there is a relation $v_1-v_2=u$ with $v_1, v_2\varepsilon D\frown D_0{}^C$, so that we must have $v_1=u+v_2\ \varepsilon D_0$, which is a cotradicton.

Therefore $\bar{G}\frown D_0{}^C$ is empty. Applying Theorem 5, we find a linear functional $f(x)\varepsilon P$ such that $f(\bar{G})=0$.

Next, we shall prove that the thus obtained $f(x) \in P$ is a functional of $\mathfrak{P}$, *i. e.* $f(x^*x) \geqq 0$, for every $x \in R$. Every x^*x is expressed in the form $x^*x = u - v$, u, $v \in D$ and $u.v = 0$.

Case (i). $u = 0$. We have $x^*x = -v$, so that $f(x^*x) = 0$ by the definition.

Case (ii). $v = 0$. We have $x^*x = u$. so that $x^*x \in D$ and $f(x^*x) \geqq 0$.

Case (iii). $u, v \neq 0$. It is easy to see, by means of the structure theorem, that there is, for every $k > 0$, a positive element q_k which commutes with x^*x, u and v, and such that $q_k x^*x\, q_k = -v\, q_k^2$. Then we have $f(v(1-q_k^2)) \leq \frac{1}{k}$, and $f(v) = f(v(1-q_k^2)) + f(v\, q_k^2) = f(v(1-q_k^2)) \leq \frac{1}{k}$, since $f(vq_k^2) = 0$ by the definition of $f(x)$. So we have $f(v) = 0$, and $f(x^*x) = f(u) \geqq 0$.

Theorem 6. For the set $\mathfrak{P}$ of all positive linear functionals on any B-algebra R with the unit element e to be not empty, it is necessary and sufficient that in R the condition (III) is satisfied.*

Proof. The sufficiency of (III) are shown in Lemma 10. The necessity is evident.

Theorem 7. The necessary and sufficient condition for $P = \mathfrak{P}$ to be valid for any B-algebra R with the unit element e is the following condition:*

(IV) $x^*x + y^*y + \cdots + z^*z = 0$ *implies* $x = y = \cdots = z = 0$,

for every $x, y, \cdots, z \in R$.

Proof. Suppose that (IV) is satisfied in R. It is clear that the condition (II) is valid, and that every $x^*x = u$, for some $u \in D$. Thus, $f(x) \in P$ implies $f(x^*x) = f(u) \geqq 0$, which shows $P < \mathfrak{P}$ As $\mathfrak{P} < P$ is evident, we have $\mathfrak{P} = P$.

Conversely, suppose that $P < \mathfrak{P}$ and there exists a relation $x^*x + y^*y + \cdots + z^*z = 0$, where $x \neq 0$, say. Put $x^*x = u - v$, where $u, v \in D$ and $u \cdot v = 0$. After two-sided multiplication by some adequate elements, we obtain another relation $x_1^*x_1 + y_1^*y_1 + \cdots + z_1^*z_1 = 0$, where $x_1^*x_1 = -v_1$, $v_1 \in D \frown D_0^c$.

By Theorem 5, Cor. 2, we have an $f(x) \in P$ with $f(v_1) > 0$. Since $P < \mathfrak{P}$ implies $f(x_1^*x_1) \geqq 0$, we have a contradiction. Thus v_1, and v, must be equal to 0. In the same manner for $y, \cdots, z$, we obtain $x = y = \cdots = z = 0$. This proves the necessity of (IV). [2]

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2) I. Kaplansky have shown that every primitive B*-algebra with a minimal right ideal is a C*-algebra. This follows from our Theorem 7. (See Lemma 4.2 in [2]). Cf. [4], also.

ON SOME CHARACTERIZATIONS OF ABSTRACT EUCLIDEAN SPACES BY PROPERTIES OF ORTHOGONALITY

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1. Introducuion. Let **B** be a real- or complex-linear normed space. If there exists a real or complex inner product in **B** such that $\| x \|^2 = (x, x)$ for every x in **B**, then **B** is called a real or complex abstract Euclidean space.

In a linear normed space **B**, four types of orthogonality can be considered. (Cf. [1] and [2].[1]) It was investigated by R. C. James [1] [2] [3] and M. M. Day [4] to characterize the abstract Euclidean space by the properties of these types of orthogonality in the linear normed space **B**.

In this paper, we shall show that the abstract Euclidean space may be characterized by relations between the above four types of orthogonality. The four types of orthogonality in linear normed spaces is as follows;

(1). *Definition of isosceles orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+y \| = \| x-y \|$.

(2). *Definition of Roberts' orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+ky \| = \| x-ky \|$ for all real numbers k.

(3). *Definition of Pythagorean orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+y \|^2 = \| x \|^2 + \| y \|^2$.

(4). *Definition of Birkhoff's orthogonality:* An element x of **B** is orthogonal to an element y if and only if $\| x+ky \| \geqq \| x \|$ for all real numbers k.

The above types of orthogonaiity are equivalent for abstrabt Euclidean spaces, but are not so in a general linear normed space. Properties of the four types of orthogonality in linear normed spaces were studied by R. C. James [1] [2].

2. Characterizations by relations between isosceles orthogonality and Birkhoff's orthogonality.

Theorem 1. *If isosceles orthogonality implies Birkhoff's orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. It will be shown that the condition given by E. R. Lorch [5] for the existence of a real inner product in a real-linear normed space is satisfied, namely:

$$\| kx+k^{-1}y \| \geqq \| x+y \|, \tag{2.1}$$

for all real numbers $k \neq 0$ and elements x and y for which $\| x \| = \| y \|$.

Suppose that $x, y \in \mathbf{B}$ and $\| x \| = \| y \|$, and set $x' = (x+y)/2$, $y' = (x-y)/2$. Then we have: $\| x'+y' \| = \| x'-y' \|$. Therefore by our condition,

$$\| x'+((k^2-1)/(k^2+1))\, y' \| \geqq \| x' \|$$

1) Numbers in brackets refer to the bibliography at the end of the paper.

for all real numbers k. From this inequality, we can derive the following inequality by using that $2|k|/(k^2+1) \geqq 1$ for all real numbers k;

$$\| k(x'+y') + k^{-1}(x'-y') \| \geqq \| 2x' \|$$

for all real numbers $k \neq 0$. Thus it is shown that the condition (2.1) is satisfied. Therefore, if **B** is a real-linear space, then **B** is a real abstract Euclidean space.

In the case that **B** is a complex-linear space, we consider the associated real-linear space **A** of **B**. (Cf. [4] pp. 334.) Then the condition (2.1) holds in **A**. Hence, by Theorem 7.2 in [4], **B** is a complex abstract Euclidean space.

Theorem 2. *If Birkhoff's orthogonality implies isosceles orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. We prove that the condition in Theorem 1 is satisfied, namely: $\| x+y \| = \| x-y \|$ implies $\| x+ky \| \geqq \| x \|$, for all real numbers k. Suppose that (2.2): $\| x+y \| = \| x-y \|$. Since the function $f(k) = \| x+ky \|$ is continuous and non-negative in real k, it must take its minimum, say at k_0. If $\| x+k_0 y \| = \min_k \| x+ky \|$, then we have:

$$\| (x+k_0 y)+(k-k_0 y) \| \geqq \| x+k_0 y \|,$$

for all real numbers k. Hence by our condition,

$$\| x+(k_0+1)y \| = \| x+(k_0-1)y \|. \tag{2.3}$$

If $|k_0| \geqq 1$, then $\| x+k_0 y \| \geqq \| x \|$. (Cf. Theorem 4.1 in [1].) But $\| x+k_0 y \| \leqq \| x \|$. Hence we have: $\| x+k_0 y \| = \| x \|$.

If $|k_0| < 1$, then from (2.2) and (2.3) we have:

$$\| x+y \| = \| x+(k_0+1)y \| = \| x+(k_0-1)y \| = \| x-y \|.$$

Since the function $f(k) = \| x+ky \|$ is convex, from the above equalities, we have

$$\| x+k_0 y \| = \| x \|.$$

Thus $\| x+ky \| \geqq \| x+k_0 y \| = \| x \|$ for all real numbers k. Hence, by Theorem 1, **B** is an abstract Euclidean space. Q.E.D.

From Theorem 2, we have the following

Corollary 1. *A real- or complex-linear normed space* **B** *is an abstract Euclidean space if and only if Birkhoff's orthogonality implies Roberts' orthogonality.*

3. Characterizations by relations between Pythagorean orthogonaliy and Birkhoff's orthogonality.

Theorem 3. *If Pythagorean orthogonality implies Birkhoff's orthogonality in a real- or copmlex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

At first, we shall prove two lemmas.

Lemma1. 1. *If* $\| x+y \| = \| x \| + \| y \|$, *and if* l, m *are real numbers such that* $lm \geqq 0$, *then* $\| lx+my \| = |l| \, \| x \| + |m| \, \| y \|$.

Proof. Suppose that $\| x+y \| = \| x \| + \| y \|$. Then,

$$\| (1-k)x+ky \| = (1-k) \| x \| + k \| y \|, \quad \text{for } 0 \leqq k \leqq 1.$$

(Cf. Proof of Theorem 4.3 in [2]). Hence

$\| x+ (k/(1-k)) y \| = \| x \| + (k/(1-k)) \| y \|$, for $0 \leqq k < 1$.
Hence, $\| x+k' y \| = \| x \| + k' \| y \|$, for all $k' \geqq 0$.

From this equality, we have Lemma 1.

Lemma 2. *If* **B** *is a real-linear space, and if Pythagorean orthogonality implies Birkhoff's orthogonality, then* **B** *is strictly convex.*

Proof. Set $x' = \| y \| x + \| x \| y$, $y' = \| y \| x$. Now, there exists a real number a such that

(3.1) $\| (a+1)x'+y' \|^2 = \| x' \|^2 + \| ax'+y' \|^2$.

(Cf. Corollary 5.1 in [1]). Therefore, from our assumption,

(3.2) $\| x'+k(ax'+y') \| \geqq \| x' \|$,

for all real numbers k.

Now, we suppose $a \geqq 0$. Then from Lemma 1, we have:

$$\| (a+1)x'+y' \| = (2a+3) \| x \| \| y \|, \quad \| ax'+y' \| = (2a+1) \| x \| \| y \|.$$

Therefore from the equality (3.1), $a = -1/2$. This contradict to our assumption that $a \geqq 0$. Therefere a must be negative.

Now, we set $k = -1/a$ in (3.2). Then $| a | \leqq \| y' \| / \| x' \|$. Since $\| x' \| = 2 \| x \| \| y \|$ from Lemma 4.1, and since $\| y' \| = \| x \| \| y \|$, we have $| a | \leqq 1/2$. Moreover, we set $k = -1$ in (3.2). Then $\| (a-1) x'+y' \| \geqq \| x' \|$. Since $\| (a-1)x'+y' \| = (1-2a) \| x \| \| y \|$, $\| x' \| = 2. \| x \| \| y \|$ by Lemma, 4.1, we have $a \leqq -1/2$. On the other hand $a \geqq -1/2$. Hence $a = -1/2$. Therefore, $\| (a+1) x'+y' \| = 2 \| x \| \| y \| = \| x' \|$. Accordingly, from the equaton (3.1), we have $(-1/2) x'+y' = 0$. Therefore, we have $y = (\| y \| / \| x \|) x$. Hence, **B** is strictly convex.

Proof of Theorem 3. (*i*). *The case that* **B** *is a real-linear space.* We suppose that $\| y+ kx \| \geqq \| y \|$ for all real numbers k. Now, there exists a number a such that

(3.3) $\| x+ (ax+y) \|^2 = \| x \|^2 + \| ax+y \|^2$.

(Corollary 5.1 in [1]). Therefore by our condition in Theorem 3, $\| (ax+y)+kx \| \geqq \| ax+y \|$ for all real numbers k. That is, y is orthogonal to x in the sense of Birkhoff, and simultaneously, $ax+y$ is also to x. Hence by Lemma 2, $a = 0$. (Cf. Theorem 4.3 in [2].) Therefore from the equality (3.3), we have $\| x+y \|^2 = \| x \|^2 + \| y \|^2$. That is:

(3.4) $\| x+y \|^2 = \| x \|^2 + \| y \|^2$

whenever $\| y+kx \| \geqq \| y \|$ for all real numbers k. From our condition in Theorem 3 and (3.4), we can see that the Pythagorean orthogonality is homogeneous. Therefore **B** is an abstract Euclidean space. (Theorem 5.2 in [1].)

(*ii*). *The case that* **B** *is a complex-linear space.*

The condition in Theorem 3 holds in the associated real-linear space **A** of **B**. Therefore **A** is a real abstract Euclidean space. Hence, by Theorem 7.2 in [4], **B** is a complex abstract Euclidean space . Q. E. D.

If $\| x+ky \| = \| x-ky \|$ for all real numbers k, then

$$\| x \| \leqq (1/2) \| x+ky \| + (1/2) \| x-ky \| = \| x+ky \|$$

for all real numbers k. Therefore, from Theorem 3, we have

Corollary 2. *If Pythagorean orthogonality implies Roberts' orthogonality in a real- or com-*

plex-linear normed space, then **B** *is an abstract Euclidean space.*

Theorem 4. *If Birkhoff's orthogonality implies Pythagorean orthogonality in a real- or complex-linear normed space* **B**, *then* **B** *is an abstract Euclidean space.*

Proof. We suppose that the condition in Theorem 3 is satisfied, namely $\|x+y\|^2=\|x\|^2+\|y\|^2$ implies $\|x+ky\|\geqq\|x\|$ for all real numbers k. Suppose that

$$\|x+y\|^2=\|x\|^2+\|y\|^2. \tag{3.5}$$

Since the function $f(k)=\|x+ky\|$ is continuous in k, and non-negative, it must take its minimum, say at k_0. If $\|x+k_0y\|=\min_k\|x+ky\|$, then $\|(x+k_0y)+((k-k_0)/\lambda)(\lambda y)\|\geqq\|x+k_0y\|$ for all real numbers k and $\lambda\neq0$. Therefore, according to our condition, $\|(x+k_0y)+\lambda y\|^2=\|x+k_0y\|^2+\|\lambda y\|^2$ for all real numbers $\lambda\neq0$. When $\lambda=0$, this equality is also true.

Therefore

$$\|x+(k_0+\lambda)y\|^2=\|x+k_0y\|^2+\lambda^2\|y\|^2$$

for all real numbers λ. By setting $\lambda=-k_0$, $1-k_0$ respectively,

$$\|x\|^2=\|x+k_0y\|^2+k_0^2\|y\|^2,\quad \|x+y\|^2=\|x+k_0y\|^2+(1-k_0)^2\|y\|^2.$$

Therefore according to the equation (3.5), we have $k_0=0$. Hence $\|x+ky\|\geqq\|x\|$ for all real numbers k.

Therfore, by Theorem 3, **B** is an abstract Euclidean space. Q.E.D.

Remark. It was proved by M.M. Day [4] that abstract Euclidean spaces are characterized by relations between isosceles orthogonality and Pythagorean orthogonality. Moreover, it was proved by E. R. Lorch [5] that abstract Euclidean spaces are characterized by the condition that *isosceles orthogonality implies Roberts' orthogonality*.

Furthermore, each of the following conditions are not sufficient in order that a real- or complex-linear normed space **B** should be an abstract Euclidean space:

(1). Roberts' orthogonality implies isosceles orthogonality.

(2). Roberts' orthogonality implies Birkhoff's orthogonality.

As to the case that Roberts' orthogonality implies Pythagorean orthogonality, no result is known.

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HOMOGENEOUS CONTACT TRANSFORMATIONS IN A GENERALIZED SPACE K_n

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1. Contact transformation. We consider the underlying n-dimensional manifofold X_n in which each set of n independent real variables $(\xi, {}^1\xi, {}^2\cdots\cdots, {}^n\xi)$ may be considered as the coordinates of a point, and then we associate a system of plane-elements π_α $(\alpha=1, 2, \cdots\cdots, n)$ to every point in X_n. Under K_n we understand the *associated* point set. The point in X_n is called the fundamental point in K_n.

At an arbitrary fundamental point P(ξ), a linear homogeneous equation $\pi_\alpha d\xi_\alpha = 0$ of $d\xi$'s with coefficients π's defines the (n−1)-direction at the point, and so π_α may be considered a plane-element at P(ξ).

A homogeneous contact transformation is defined by the following equations:

$$(1.1)\qquad \begin{cases} x^i = x^i(\xi^1, \cdots\cdots, \xi^n; \pi_1, \cdots\cdots, \pi_n), \\ p_i = p_i(\xi^1, \cdots\cdots, \xi^n; \pi_n, \cdots\cdots, \pi \\ p_i\, dx^i = \pi_\alpha d\xi^\alpha, \end{cases}$$

where x^i and p^i are homogeneous in the π's of degree zero and one respectivel y[1].

A neccessary and sufficient condition that equation (1.1) holds for arbitrary values of $d\xi^\alpha$ and $d\pi^\alpha$ is

$$(1.2)\qquad p_i\, \partial x^i/\partial\xi^\alpha = \pi_\alpha, \qquad p_i\, \partial x^i/\partial\pi_\alpha = 0.$$

While, in accordance with Euler's formula, we have

$$(1.3)\qquad \pi_\alpha\, \partial x^i/\partial\pi^\alpha = 0, \qquad \pi_\alpha\, \partial p_i/\partial\pi_\alpha = p_i.$$

We also have

$$\partial p_i/\partial\xi^\beta\ \partial x^i/\partial\xi^\alpha - \partial p_i/\partial\xi^\alpha\ \partial x^i/\partial\xi^\beta = 0,$$

$$(1.4)\qquad \partial p_i/\partial\pi_\beta\ \partial x^i/\partial\xi^\alpha - \partial p_i/\partial\xi^\alpha\ \partial x^i/\partial\pi_\beta = \delta^\beta_\alpha,$$

$$\partial p_i/\partial\pi_\beta\ \partial x^i/\partial\pi_\alpha - \partial p_i/\partial\pi_\alpha\ \partial x^i/\partial\pi_\beta = 0.$$

The unique inverse of a contact transformation (1.1) is

$$(1.5)\qquad \begin{cases} \xi^\alpha = \xi^\alpha(x^1, \cdots\cdots, x^n; p_1, \cdots\cdots, p_n), \\ \pi_\alpha = \pi_\alpha(x^1, \cdots\cdots, x^n; p_1, \cdots\cdots, p_n), \\ p_i\, dx^i = \pi_\alpha\, d\xi^\alpha, \end{cases}$$

and analogously to (1.2) and (1.3) we have

$$(1.6)\qquad \pi_\alpha \partial\xi^\alpha/\partial x^i = p_i, \qquad \pi_\alpha \partial\xi^\alpha/\partial p_i = 0$$

and

$$(1.7)\qquad p_i\, \partial\xi^\alpha/\partial p_i = 0, \qquad p_i\, \partial\pi_\alpha/\partial p_i = \pi_\alpha.$$

We have also the following identities frequently used in this paper:

$$\partial\xi^\alpha/\partial x^i = \partial p_i/\partial\pi_\alpha, \quad \partial\pi_\alpha/\partial p_i = \partial x^i/\partial\xi^\alpha,$$
$$\partial\xi^\alpha/\partial p_i = -\partial x^i/\partial\pi_\alpha, \quad \partial\pi_\alpha/\partial x^i = -\partial p_i/\partial\xi^\alpha, \tag{1.8}$$

by which we derive from (1.4) the following:

$$\partial\pi_\alpha/\partial p_i\,\partial\pi_\beta/\partial x^i - \partial\pi_\alpha/\partial x^i\,\partial\pi_\beta/\partial p_i = 0,$$
$$\partial\pi_\alpha/\partial p_i\,\partial\xi^\beta/\partial x^i - \partial\pi_\alpha/\partial x^i\,\partial\xi^\beta/\partial p_i = \delta_\alpha^\beta, \tag{1.9}$$
$$\partial\xi^\alpha/\partial p_i\,\partial\xi^\beta/\partial x^i - \partial\xi^\alpha/\partial x^i\,\partial\xi^\beta/\partial p_i = 0.$$

Next we shall consider infinitesimal homogeneous contact transformations. Each infinitesimal homogeneous contact transformation is defined by equations of the form

$$x^i = \xi^i + \partial C/\partial\pi_i\,\delta t, \qquad p_i = \pi_i - \partial C/\partial\xi^i\,\delta t, \tag{1.10}$$

where C is homogeneous of degree one in the π's, moreover any such function C determines an infinitesimal homogeneous contact transformation. The function C is called the characteristic function of the transformation.

If we form the differentials of (1.10), we obtain

$$dx^i = d\xi^i + \left(\frac{\partial^2 C}{\partial\pi_i\,\partial\xi^j}\,d\xi^j + \frac{\partial^2 C}{\partial\pi_i\,\partial\pi_j}\,d\pi_j\right)\delta t,$$
$$dp_i = d\pi_i - \left(\frac{\partial^2 C}{\partial\xi^i\,\partial\xi^j}\,d\xi^j + \frac{\partial^2 C}{\partial\xi^i\,\partial\pi_j}\,d\pi_j\right)\delta t.$$

These equations and (1.10) define the extended infinitesimal transformation. Hence in accordance with the general theory of continuous groups, the quantities $\pi_i\,d\xi^i$ is invariant under the finite guoup G_1 generated by the extended infinitesimal transformation, and the finite equations of G_1 are given by the integral of the equations

$$d\xi^i/dt = \partial C/\partial\pi_i, \qquad d\pi_i/dt = -\partial C/\partial\xi^i, \tag{1.11}$$

say

$$x^i = \varphi^i(\xi, \pi, t), \qquad p_i = \psi_i(\xi, \pi, t), \tag{1.12}$$

and their differentials. Conversely (1.12) define a one-parameter group of contact transformations.

If we transform the equations (1.11) by means of a general homogeneous contact transformation (1.1), we have

$$dx^i/dt = \partial\bar{C}/\partial p_i, \qquad dp_i/dt = -\partial\bar{C}/\partial x^i. \tag{1.13}$$

Hence we have:

A group G_1 of homogeneous contact transformations is transformed into another group G_1 by any homogeneous contact transformation and the equations of the new group are the integrals of the equation (1.13), where $\bar{C}$ is the transform of the characteristic function of the given group.

Putting

$$2\ H(\xi, \pi) = C^2(\xi, \pi),$$

now we shall adopt the following as the contravariant fundamental tensor

$$\gamma^{\alpha\beta} = \frac{\partial^2 H}{\partial\pi_\alpha\, \partial\pi_\beta}, \tag{1.14}$$

which introduces metric properties in our space K_n and we define the covariant fundamental tensor by means of

$$\gamma^{\alpha\beta}\ \gamma_{\beta\gamma} = \delta^\alpha_\gamma.$$

The homogeneity property of C implies the following:

$$\begin{cases} \dfrac{\partial^2 H}{\partial\pi_\alpha\, \partial\pi_\beta}\,\pi_\alpha\,\pi_\beta = 2\ H \ \textit{or}\ \ \gamma^{\alpha\beta}\,\pi_\alpha\,\pi_\beta = 2\ H, \\ \dfrac{\partial^2 H}{\partial\pi_\alpha\, \partial\pi_\beta}\,\pi_\beta = C\dfrac{\partial C}{\partial\pi_\alpha} \ \textit{or}\ \ \gamma^{\alpha\beta}\,p_\beta = C\ \dfrac{\partial C}{\partial\pi_\alpha}, \end{cases} \tag{1.15}$$

$$\begin{cases} l_\alpha = \dfrac{\pi_\alpha}{C} = \gamma_{\alpha\beta} l^\beta, \\ l^\alpha = \partial C/\partial\pi_\alpha = \gamma^{\alpha\beta}\, l_\beta. \end{cases} \tag{1.16}$$

From the above equation (1.13) we see that the magnitude of the covariant vector π_α is C and $C\partial C/\partial\pi_\alpha$ is a contravariant vector perpendicular to the plane-element π_α and moreover π_α/C and $\partial C/\partial\pi_\alpha$ are both unit vectors.

2. Linear displacements in K_n. Now we shall define the absolute differentials of vectors, when a fundamental point $P(\xi)$ is displaced to a near point $(\xi + d\xi)$ and its plane-element π_α at the point P is changed slightly, say, into $\pi_\alpha + d\pi_\alpha$, by the following equations,

$$\begin{cases} \delta v^\alpha = dv^\alpha + C_\beta^{\alpha\gamma}\ v^\beta\, d\pi_\gamma + \Gamma^\alpha_{\beta\gamma}\, v^\beta\, d\xi^\gamma, \\ \delta v_\alpha = dv_\alpha - C_\alpha^{\beta\gamma}\, v_\beta\, d\pi_\gamma - \Gamma^\beta_{\alpha\gamma}\, v_\beta\, d\xi^\gamma. \end{cases} \tag{2.1}$$

1°. *From the standpoint of contact transformation a change $\pi_\alpha \to \rho\pi_\alpha$ is of no significance, so that, the functions $\Gamma^\alpha_{\beta\gamma}(\xi, \pi)$ and $C^{\alpha\gamma}_\beta(\xi, \pi)$ may be assumed to be homogeneous of degree zero and one in the π's respectively, and the relations $C^{\beta\gamma}_\alpha\,\pi_\gamma = 0$ may be also to hold good.*

2°. *Secondly we assume that our connection is a metric one. This assumption leads to the following*

$$\partial\gamma_{\alpha\beta}/\partial\xi^\gamma = \gamma_{\tau\beta}\ \Gamma^\tau_{\alpha\gamma} + \gamma_{\tau\alpha}\ \Gamma^\tau_{\beta\gamma}, \qquad \partial\gamma_{\alpha\beta}/\partial\pi_\gamma = \gamma_{\tau\alpha}\ C^{\tau\gamma}_\beta + \gamma_{\tau\beta}\ C^{\tau\gamma}_\alpha, \tag{2.2}$$

or

$$\partial\gamma_{\alpha\beta}/\partial\xi^\gamma = \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma}, \qquad \partial\gamma_{\alpha\beta}/\partial\pi_\gamma = C^{\ \gamma}_{\beta\alpha} + C^{\ \gamma}_{\alpha\beta}, \tag{2.3}$$

and

$$\partial\gamma^{\alpha\beta}/\partial\xi_\gamma = -\Gamma^{\beta\alpha}_{\ \ \gamma} - \Gamma^{\alpha\beta}_{\ \ \gamma}, \qquad \partial\gamma^{\alpha\beta}/\partial\pi_\gamma = -C^{\beta\alpha\gamma} - C^{\alpha\beta\gamma}. \tag{2.4}$$

3°. *Denote X and Y two vectors of the same linear element (ξ, π); let DX and DY their absolute differentials when their contravariant components X^i and Y^i are fixed and when*

their common linear element (ξ, π) *rotate infinitesimally about their center. Then we assume that*

$$X \cdot \bar{D}Y = Y \cdot \bar{D}X.$$

The law of symmetry implies

$$C^{\alpha\beta\gamma} = C^{\beta\alpha\gamma}.$$

Combining the above relations with $(2.4)_2$ we have

$$C^{\alpha\beta\gamma} = -\frac{1}{2}\,\partial\gamma^{\alpha\beta}/\partial\pi_\gamma \text{ or } C^{\gamma}_{\alpha\beta} = \frac{1}{2}\,\partial\gamma_{\alpha\beta}/\partial\pi_\gamma. \tag{2.5}$$

For arbitrary function $f(\xi, \pi)$ we put

$$f^{\alpha}_{\parallel} = \partial f/\partial\pi_\alpha\, C \tag{2.6}$$

and for arbitrary quantities, say $T_{\alpha\beta}$, we denote as follows

$$T_{\alpha\beta}\, l^\beta = T_{\alpha 0}, \quad \textit{etc.} \tag{2.7}$$

Putting

$$A^{\alpha\beta\gamma} = C\, C^{\alpha\beta\gamma} = -\frac{1}{2}\,\gamma^{\alpha\beta}{}_{\parallel}{}^{\gamma}, \tag{2.8}$$

the tensor $A^{\alpha\beta\gamma}$ is symmetric and the contracted tensor $A^{o\alpha\beta}$ is a zero tensor.

The absolute differential of any vector V^α may be written as follows:

$$Dv^\alpha = dv^\alpha + v^\beta \Gamma^{\alpha}_{\beta\gamma}\, d\xi^\gamma + v^\beta C^{\alpha\gamma}_{\beta}\, d\pi_\gamma, \tag{2.9}$$

and for l^α and l_α we have respectively

$$Dl^\alpha = dl^\alpha + l^\beta \Gamma^{\alpha}_{\beta\gamma}\, d\xi^\gamma, \text{ or } dl^\alpha = Dl^\alpha - \Gamma^{\alpha}_{o\gamma}\, d\xi^\gamma \tag{2.10}$$

and

$$Dl_\alpha = dl_\alpha - l_\beta \Gamma^{\beta}_{\alpha\gamma}\, d\xi^\gamma, \quad \text{or } dl_\alpha = Dl_\alpha + \Gamma^{o}_{\alpha\gamma}\, d\xi^\gamma. \tag{2.11}$$

Substituting (2.11) into (2.9) we obtain

$$Dv^\alpha = dv^\alpha + v^\beta \Gamma^{*\alpha}_{\beta\gamma}\, d\xi^\gamma + v^\beta\, A^{\alpha\gamma}_{\beta}\, Dl_\gamma, \tag{2.12}$$

where

$$\Gamma^{*\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + A^{\alpha\tau}_{\beta}\, \Gamma_{\tau o\gamma}. \tag{2.13}$$

By means of the above relations, (2.10) and (2.11) become

$$Dl^\alpha = dl^\alpha + \Gamma^{*\alpha}_{o\gamma}\, d\xi^\gamma \tag{2.14}$$

and

$$Dl_\alpha = dl_\alpha - \Gamma^{*}_{\alpha o\gamma}\, d\xi^\gamma \tag{2.15}$$

respectively.

4°. *Finally we assume the law of symmetry:*

$$\Gamma^{*}_{\alpha\beta\gamma} = \Gamma^{*}_{\gamma\beta\alpha}. \tag{2.16}$$

We have easily the following relations:

(2.17) $$\Gamma^{*}_{\alpha\beta\gamma} + \Gamma^{*}_{\beta\alpha\gamma} = \partial\gamma_{\alpha\beta}/\partial\xi^{\gamma} + 2A_{\alpha\beta}^{\tau}\Gamma_{\tau 0\gamma},$$

(2.18) $$\Gamma^{*}_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\alpha\beta}^{\tau}\Gamma_{\tau 0\gamma} - A_{\alpha\gamma}^{\tau}\Gamma_{\tau 0\beta},$$

and

(2.19) $$\Gamma_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}\Gamma_{\tau 0\alpha} - A_{\alpha\gamma}^{\tau}\Gamma_{\tau 0\beta}.$$

contracting (2.19) by l^{β}, we have

(2.20) $$\Gamma_{\alpha 0\gamma} = [\alpha\gamma, 0] - A_{\alpha\gamma}^{\tau}\Gamma_{\tau 00} = [\alpha\gamma, o] - A_{\alpha\gamma}^{\tau}[\tau o, o],$$

because

(2.21) $$\Gamma_{\gamma 00} = [\gamma o, o].$$

Substituting (2.20) into (2.19) or (2.18), we have

(2.22) $$\Gamma_{\alpha\beta\gamma} = [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}([\tau\alpha, o] - A_{\tau\alpha}^{\sigma}[\sigma o, o]) - A_{\alpha\gamma}^{\tau}([\tau\beta, o] - A_{\tau\beta}^{\sigma}[\sigma o, o]),$$

or

(2.23) $$\begin{aligned}\Gamma^{*}_{\alpha\beta\gamma} = {} & [\alpha\gamma, \beta] + A_{\gamma\beta}^{\tau}([\tau\alpha, o] - A_{\tau\alpha}^{\sigma}[\sigma o, o]) \\ & + A_{\alpha\beta}^{\tau}([\tau\gamma, o] - A_{\tau\gamma}^{\sigma}[\sigma o, o]) - A_{\alpha\gamma}^{\tau}([\tau\beta, o] - A_{\tau\beta}^{\sigma}[\sigma o, o]).\end{aligned}$$

Thus we have determined the connection completely.

Consider a variable vector, for example X^{α}. Since $\partial X^{\alpha}/\partial\pi_{\tau}\ \pi_{\tau} = 0$, its absolute differential can be put in the form

(2.24) $$DX^{\alpha} = X^{\alpha}|_{\gamma}\, d\xi^{\gamma} + X^{\alpha}|^{\gamma}\, Dl_{\gamma},$$

where

(2.25) $$X^{\alpha}|_{\gamma} = \partial X^{\alpha}/\partial\xi^{\gamma} + X^{\alpha}\|^{\beta}\Gamma^{*}_{\beta 0\gamma} + X^{\beta}\Gamma^{*\alpha}_{\beta\gamma}, \quad X^{\alpha}|^{\gamma} = X^{\alpha}\|^{\gamma} + X^{\beta}A^{\alpha\gamma}_{\beta}.$$

Particularly for the fundamental tensor, we have

$$\gamma_{\alpha\beta}|_{\gamma} = \gamma^{\alpha\beta}|_{\gamma} = \gamma_{\alpha\beta}|^{\gamma} = \gamma^{\alpha\beta}|^{\gamma} = 0.$$

3. Spaces $\mathbf{T}_n$ as transforms of spaces $\mathbf{K}_n$ [2]. Consider the identity

(3.1) $$P(x, p) = H(\xi, \pi)$$

under a homogeneous contact transformation. We notice that the function H is homogeneous of degree two in the π's and that the function P is homogeneous of degree two in the p's. By differentiation we have

(3.2) $$\begin{aligned}\partial P/\partial p_i &= \partial H/\partial\pi_{\alpha}\,\partial\pi_{\alpha}/\partial p_i + \partial H/\partial\xi_{\alpha}\,\partial\xi^{\alpha}/\partial p_i, \\ \partial P/\partial x^i &= \partial H/\partial\pi_{\alpha}\,\partial\pi_{\alpha}/\partial x^i + \partial H/\partial\xi_{\alpha}\,\partial\xi^{\alpha}/\partial x^i,\end{aligned}$$

or

(3.2)′ $$\begin{aligned}\partial P/\partial p_i &= \partial H/\partial\pi_{\alpha}\,\partial x^i/\partial\xi_{\alpha} - \partial H/\partial\xi^{\alpha}\,\partial x^i/\partial\pi_{\alpha}, \\ -\partial P/\partial x^i &= \partial H/\partial\pi_{\alpha}\,\partial p_i/\partial\xi^{\alpha} - \partial H/\partial\xi^{\alpha}\,\partial p_i/\partial\pi_{\alpha}.\end{aligned}$$

From homogeneity property of H in the π's, when we define quantities ξ'^{α} by $\xi'^{\alpha} = \partial H/\partial \pi_{\alpha}$, we have

$$\xi'^{\alpha} = \partial H/\partial \pi_{\alpha} = \gamma^{\alpha\beta}\pi_{\beta}, \tag{3.3}$$

$$\pi_{\alpha} = \gamma_{\alpha\beta}\,\xi'^{\beta}, \tag{3.4}$$

and

$$H = \frac{1}{2}\gamma^{\alpha\beta}\pi_{\alpha}\pi_{\beta}, \qquad H = \frac{1}{2}\gamma_{\alpha\beta}\,\xi'^{\alpha}\,\xi'^{\beta}. \tag{3.5}$$

As we can solve (3.4) with respect to π_{α}, we can write (3.5) in terms of ξ^{α} and ξ'^{α}. When it is done we denote $\frac{1}{2}\gamma_{\alpha\beta}\,\xi'^{\alpha}\,\xi'^{\beta}$ by $\phi(\xi, \xi')$ as follows:

$$\phi(\xi, \xi') = \frac{1}{2}\gamma_{\alpha\beta}\,\xi'^{\alpha}\,\xi'^{\beta}. \tag{3.6}$$

From (3.3) and (3.6), we can introduce the following relations

$$\gamma_{\alpha\beta} = \partial\pi_{\beta}/\partial\xi'^{\alpha}, \tag{3.7}$$

$$\partial\phi/\partial\xi'^{\alpha} = \pi_{\alpha}, \tag{3.8}$$

$$\frac{\partial^2\phi}{\partial\xi'^{\alpha}\,\partial\xi'^{\beta}} = \gamma_{\alpha\beta}, \tag{3.9}$$

$$\partial\gamma_{\alpha\beta}/\partial\xi_{\gamma} = -\left[\gamma^{\alpha\varepsilon}\left\{\begin{matrix}\beta\\ \varepsilon\gamma\end{matrix}\right\} + \gamma^{\varepsilon\beta}\left\{\begin{matrix}\alpha\\ \varepsilon\gamma\end{matrix}\right\}\right], \qquad \partial\gamma_{\beta\alpha}/\partial\xi^{\gamma} = [\beta\gamma, \alpha] + [\alpha\gamma, \beta]. \tag{3.10}$$

From $(3.5)_1$ and (3.10) we have

$$\partial H/\partial\xi_{\gamma} = -\,\xi'^{\delta}\xi'^{\beta}\,[\delta\gamma, \beta] \tag{3.11}$$

and from (3.6)

$$\partial\phi/\partial\xi^{\gamma} = -\,\pi_{\alpha}\,\gamma^{\alpha\delta}\pi_{\varepsilon}\left\{\begin{matrix}\varepsilon\\ \delta\gamma\end{matrix}\right\} = \xi'^{\delta}\,\xi'^{\beta}\,[\gamma\delta, \beta]. \tag{3.12}$$

Hence we have

$$\partial\phi/\partial\xi^{\gamma} = -\,\partial H/\partial\xi^{\gamma}. \tag{3.13}$$

When we define x'^{i} by the following

$$x'^{i} = \partial P/\partial p_{i}, \tag{3.14}$$

x'^{i} is homogeneous of degree one in the p's.

From the definition of ξ'^{α} the first equation of (3.2) is

$$\partial P/\partial p_{i} = \partial H/\partial\pi_{\alpha}\left(\partial x^{i}/\partial\xi_{\alpha} + \pi_{\sigma}\left\{\begin{matrix}\sigma\\ \alpha\varepsilon\end{matrix}\right\}\partial x^{i}/\partial\pi_{\varepsilon}\right) \tag{3.15}$$

in consequence of (3.11). When we define g^{ij} by

$$g^{ij} = \frac{\partial^2 P}{\partial p_{i}\,\partial p_{j}}, \tag{3.16}$$

immediately we have the following relations

$$g^{ij}\,p_{i}\,p_{j} = \gamma^{\alpha\beta}\pi_{\alpha}\pi_{\beta} = 2P. \tag{3.17}$$

Differentiating the above equation we have

$$\partial P/\partial x^k = \tfrac{1}{2}\,\partial g^{ij}/\partial x^k\, p_i p_j = -\tfrac{1}{2}\, g^{ij}\,\partial P/\partial p_h p_i \left\{[jk, h] + [hk, j]\right\}$$
$$= -\,\partial P/\partial p_i\,\partial P/\partial p_j\,[jk,\ i], \tag{3.18}$$

in concequence of the similar equations to (3.10), where g_{ij} is the inverse of g^{ij}.

On the other hand from (3.2)$_2$ and (3.11), we have

$$-\,\partial P/\partial x^k = \partial H/\partial \pi_\sigma \left(\partial p_k/\partial \xi^\sigma + \pi_\varepsilon \left\{{\varepsilon \atop \sigma\tau}\right\} \partial p_k/\partial \pi_\varepsilon\right). \tag{3.19}$$

From this result and equation (3.18), we get

$$\partial P/\partial p_i\,\partial P/\partial p_j\,[jk, i] = \partial H/\partial \pi_\sigma \left(\partial p_k/\partial \xi^\sigma + \pi_\varepsilon \left\{{\varepsilon \atop \sigma\tau}\right\} \partial p_k/\partial \pi_\tau\right). \tag{3.20}$$

From (3.18) and the similar equations to (3.2)′:

$$\partial H/\partial \pi_\alpha = \partial P/\partial p_i\,\partial \xi^\alpha/\partial x^i - \partial P/\partial x^i\,\partial \xi^\alpha/\partial p_i,$$
$$-\,\partial H/\partial \xi^\alpha = \partial P/\partial p_i\,\partial \pi_\alpha/\partial x^i - \partial P/\partial x^i\,\partial \pi_\alpha/\partial p_i, \tag{3.21}$$

we get the following relation

$$\partial H/\partial \pi_\alpha = \partial P/\partial p_i \left(\partial \xi^\alpha/\partial x^i + \partial P/\partial p_j\,[ik,\ j]\ \partial \xi^\alpha/\partial p_k\right), \tag{3.22}$$

which is the inverse of (3.15). And the inverse of (3.20) is following

$$\pi_\gamma \gamma^{\delta\gamma}\,\pi_\varepsilon\,\gamma^{\varepsilon\beta}\,[\delta\alpha,\ \beta] = -\,\partial P/\partial \xi^\alpha = \partial P/\partial p_j$$
$$\left(\partial \pi_\alpha/\partial x^j + \partial P/\partial p_k\,[ji,\ k]\,\partial \pi_\alpha/\partial p_i\right). \tag{3.23}$$

When any contact transformation is applied to our space K_n, we have spaces T_n as its transforms. Consider a non-singular transformation in K_n

$$\xi^\alpha = f^{-1\alpha}(\bar{\xi}), \qquad \pi_\alpha = \bar{\pi}_\beta\,\partial \bar{\xi}^\beta/\partial \xi^\alpha. \tag{3.24}$$

Then if the coordinates x^i in T_n are subject to the transformation

$$x^i = f^{-1i}(\bar{x}) \tag{3.25}$$

and if we put

$$p_i\,\partial x^i/\partial \bar{x}^j = \bar{p}_j, \qquad p_i = \bar{p}_j\,\partial \bar{x}^j/\partial x^i, \tag{3.26}$$

and

$$H(\xi,\ \pi) = \bar{H}(\bar{\xi},\ \bar{\pi}), \tag{3.27}$$

we have

$$p_i\,\partial \bar{x}^i/\partial \xi^\alpha = \bar{\pi}_\alpha, \qquad \bar{p}_i\,\partial \bar{x}^i/\partial \bar{\pi}_\alpha = 0 \quad \textit{and} \quad \partial H/\partial \pi_\alpha = \partial H/\partial \bar{\pi}_\beta\,\partial \xi^\alpha/\partial \bar{\xi}^\beta.$$

Hence $\partial H/\partial \pi_\alpha$ and $\partial \bar{H}/\partial \pi_\beta$ are the components of a contravariant vector in K_n in their respective coordinates. From (3.1) and (3.27) we have

$$P(x,\ p) = \bar{P}(\bar{x},\ \bar{p}). \tag{3.28}$$

Differentiating the above relation we have

$$\partial P/\partial p_i = \partial \bar{P}/\partial \bar{p}_j\,\partial x^i/\partial \bar{x}^j, \qquad \partial \bar{P}/\partial \bar{p}_i = \partial P/\partial p_j\,\partial \bar{x}^i/\partial x^j. \tag{3.29}$$

From the equations (3.24) we have

$$\partial\pi_\gamma/\partial\bar{\xi}^\delta\,\partial\xi^\gamma/\partial\bar{\xi}^\sigma + \pi_\gamma\,\partial^2\xi^\gamma/\partial\bar{\xi}^\delta\,\partial\bar{\xi}^\sigma = 0.$$

On the other hand from (2.23) we have

$$\partial^2\xi^\gamma/\partial\bar{\xi}^\delta\,\partial\bar{\xi}^\sigma + \Gamma^{*\gamma}_{\alpha\varepsilon}\,\partial\xi^\alpha/\partial\bar{\xi}^\delta\,\partial\xi^\varepsilon/\partial\bar{\xi}^\sigma = \bar{\Gamma}^{*\beta}_{\delta\sigma}\,\partial\xi^\gamma/\partial\bar{\xi}^\beta.$$

From above two equations and relations:

$$\partial\bar{x}^j/\partial\bar{\xi}^\sigma = \partial\bar{x}^j/\partial x^i\left(\partial x^i/\partial\xi^\alpha\,\partial\xi^\alpha/\partial\bar{\xi}^\sigma + \partial x^i/\partial\pi_\gamma\,\partial\pi^\gamma/\partial\bar{\xi}^\sigma\right),$$

$$\partial\bar{x}^j/\partial\bar{\pi}_\varepsilon = \partial\bar{x}^j/\partial x^i\,\partial x^i/\partial\pi_\gamma\,\partial\bar{\xi}^\varepsilon/\partial\xi^\gamma,$$

we have

$$\partial\bar{x}^j/\partial\bar{\xi}^\delta + \bar{\beta}_{\delta\varepsilon}\,\partial\bar{x}^j/\partial\bar{\pi}_\varepsilon = \partial\bar{x}^j/\partial x^i\,\partial\xi^\alpha/\partial\bar{\xi}^\delta\,f^i_\alpha, \tag{3.30}$$

where $\beta_{\alpha\gamma} = \pi^\beta\,\Gamma^{*\beta}_{\alpha\gamma}$ and $f^i_\alpha = \partial x^i/\partial\xi^\alpha + \beta_{\alpha\gamma}\,\partial x^i/\partial\pi_\gamma$.

Since

$$\partial H/\partial\pi_\mu\,\pi_\rho\,\gamma^{\rho\lambda}\,\Gamma^*_{\mu\lambda\nu} = 2H\,\Gamma^*_{00\nu} = 2H\,[0\nu, 0] = \partial H/\partial\pi_\mu\,\pi_\rho\left\{{\rho \atop \mu\nu}\right\},$$

from (3.30) we have

$$\frac{\partial\bar{H}}{\partial\pi_\delta}\left(\frac{\partial\bar{\pi}^j}{\partial\bar{\xi}^\delta} + \bar{\pi}\left\{{\bar{\rho} \atop \delta\varepsilon}\right\}\frac{\partial\bar{\pi}^j}{\partial\bar{\pi}_\varepsilon}\right) = \frac{\partial H}{\partial\pi_\alpha}\left(\frac{\partial x^i}{\partial\xi^\alpha} + \pi_\beta\left\{{\beta \atop \alpha\gamma}\right\}\frac{\partial x^i}{\partial\pi_\tau}\right)\frac{\partial\bar{x}^j}{\partial x^i} \tag{3.31}$$

by multiplying $\partial\bar{H}/\partial\bar{\pi}_\delta$. If we multiply (3.15) by $\partial\bar{x}^j/\partial x^i$ and if we compare the result with (3.31), we have

$$\frac{\partial P}{\partial p_i}\frac{\partial\bar{x}^j}{\partial x^i} = \frac{\partial\bar{H}}{\partial\bar{\pi}_\delta}\left(\frac{\partial\bar{x}^j}{\partial\bar{\xi}^\delta} + \bar{\pi}_\rho\left\{{\bar{\rho} \atop \delta\varepsilon}\right\}\frac{\partial\bar{x}^j}{\partial\bar{\pi}_\varepsilon}\right) = \frac{\partial\bar{P}}{\partial\bar{p}_j}. \tag{3.32}$$

4. Tensors in spaces T_n. Let $X^\alpha(\xi, \pi)$ which are homogeneous of degree zero in the π's, be the components of a contravariant vector in K_n and define functions $v^i(x, p)$ in T_n by the equations

$$v^i = f^i_\alpha\,X^\alpha. \tag{4.1}$$

Let $X_\alpha(\xi, \pi)$ which are homogeneous of degree zero in the π's, be the components of a contravariant vector in K_n and define functions $v_i(x, p)$ in T_n by the equations

$$v_i = X_\alpha\left(\partial\xi^\alpha/\partial x^i + b_{ij}\,\partial\xi^\alpha/\partial p_j\right), \tag{4.2}$$

where the functions b_{ij} are symmetric in its indices and are homogeneous of degree one in the p's and are to be such that

$$\partial p_i/\partial\xi^\alpha = \partial x^j/\partial\pi_\sigma\,\beta_{\sigma\alpha}\,b_{ij} - \partial p_i/\partial\pi_\sigma\,\beta_{\sigma\alpha} + \partial x^j/\partial\xi_\alpha\,b_{ij}. \tag{4.3}$$

Then by the equation (4.3) we have

$$f^i_\alpha\,h^\beta_i = \delta^\beta_\alpha, \tag{4.4}$$

where we put

$$h^\beta_i = \partial\xi^\beta/\partial x^i + b_{ij}\,\partial\xi^\beta/\partial p_j.$$

Therefore by means of (4.1), (4.2) and (4.4) we have

$$v_i v^i = X_\alpha X^\alpha .$$

Since

$$\partial x^i / \partial \xi^\alpha \, \partial \xi^\beta / \partial x^i = \delta^\beta_\alpha - \partial p_j / \partial \xi^\alpha \, \partial \xi^\beta / \partial p_j ,$$

$$\partial x^i / \partial \pi_\gamma \, \partial \xi^\beta / \partial x^i = - \partial p_j / \partial \pi_\gamma \, \partial \xi^\beta / \partial p_j ,$$

the equation (4.3) is written in the form

$$\partial p_j / \partial \xi^\alpha + \beta_{\alpha\gamma} \partial p_j / \partial \pi_\gamma - b_{ij} \left(\partial x^i / \partial \xi^\alpha + \beta_{\alpha\gamma} \partial x^i / \partial \pi_\gamma \right) = 0 \tag{4.5}$$

or

$$\partial \pi_\alpha / \partial x^j + b_{ij} \partial \pi_\alpha / \partial p_i - \beta_{\alpha\gamma} \left(\partial \xi^\gamma / \partial x^j + b_{ij} \partial \xi^\gamma / \partial p_i \right) = 0. \tag{4.6}$$

Multiplying (4.5) by $\partial H / \partial \pi_\alpha$ and making use of (3.15) and (3.20), we have

$$\frac{\partial P}{\partial p_i} \frac{\partial P}{\partial p_k} [ij, k] - b_{ij} \frac{\partial P}{\partial p_i} = 0,$$

$$i.\ e. \quad b_{ij} \frac{\partial P}{\partial p_i} = \Gamma^*_{jki} \frac{\partial P}{\partial p_i} \frac{\partial P}{\partial p_k} = \frac{\partial P}{\partial p_i} \frac{\partial P}{\partial p_k} [ij, k].$$

Hence we have

$$b_{ij} = \Gamma^*_{ikj} \frac{\partial P}{\partial p_k} + C_{ij} = \beta_{ij} + C_{ij}, \tag{4.7}$$

where

$$C_{ij} \frac{\partial P}{\partial p_i} = 0, \qquad C_{ij} = C_{ji}. \tag{4.8}$$

From (4.6) we have the equation analogous to (4.3)

$$\partial \pi_\lambda / \partial x^i = \partial \xi^\nu / \partial p_j \, b_{ij} \, \beta_{\nu\lambda} - \partial \pi_\lambda / \partial p_j \, b_{ij} + \partial \xi^\nu / \partial x^i \, \beta_{\lambda\nu}. \tag{4.9}$$

By means of (4.5) and the identities

$$\frac{\partial x^i}{\partial \xi^\alpha} \frac{\partial \xi^\alpha}{\partial x^j} + \frac{\partial x^i}{\partial \pi_\alpha} \frac{\partial \pi_\alpha}{\partial x^j} = \delta^j_i , \qquad \frac{\partial x^i}{\partial \xi^\alpha} \frac{\partial \xi^\alpha}{\partial p_j} + \frac{\partial x^i}{\partial \pi_\alpha} \frac{\partial \pi_\alpha}{\partial p_j} = 0$$

we obtain

$$f^i_\alpha \, h^\alpha_j = \delta^i_j . \tag{4.10}$$

When a transformation of coordinates (3.24), (3.25) and (3.26) is made in K_n and T_n, the equation (4.1) multiplied by $\partial \bar{x}^j / \partial x^i$ is written by means of (3.30) in the form:

$$v^i \, \partial \bar{x}^j / \partial x^i = v^\sigma \left(\partial \bar{x}^j / \partial \bar{\xi}^\sigma + \bar{\beta}_{\sigma\varepsilon} \partial \bar{x}^j / \partial \pi_\varepsilon \right) = v^j ,$$

therefore v^i, as defined by (4.1), are components of a contravariant vector in T_n. Also we see that v^i, as defined by (4.2), are components of a covariant vector in T_n. From (3.15) we have

$$\partial P / \partial p_i = \partial H / \partial \pi_\alpha \, f^i_\alpha . \tag{4.11}$$

Differentiating (4.11), and making use of the relation

(4.12) $$\partial^2 H/\partial\pi_\alpha\,\partial\xi^\varepsilon = -\left\{{\alpha \atop \delta\varepsilon}\right\}\gamma^{\tau\delta}\,\pi_\tau - \gamma^{\alpha\delta}\,\beta_{\delta\varepsilon} - A^{\alpha\ \omega}_{\ \varepsilon}\,[\omega\lambda,\,0]\,\partial H/\partial\pi_\lambda,$$

which is obtained by successive differentiation of (3.5), we have, after some calculation,

$$g^{ij} = \gamma^{\alpha\beta}\,f^i_\alpha\,f^j_\beta + \partial H/\partial\pi_\delta\Big(\partial f^i_\delta/\partial\pi_\beta\,\partial x^j/\partial\xi^\beta - \Big(\partial f^i_\delta/\partial\xi^\beta - f^i_\alpha\left\{{\alpha \atop \delta\beta}\right\} - A^{\alpha\ \omega}_{\ \beta}\,[\omega\delta,\,o]\,f^i_\alpha\Big)\,\partial x^j/\partial\pi_\beta\Big).$$

But we can prove that the second term of the second member of the above equation vanishes, hence we have

(4.13) $$g^{ij} = \gamma^{\alpha\beta}\,f^i_\alpha\,f^j_\beta \quad \textit{or} \quad g_{ij} = \gamma_{\alpha\beta}\,h^\alpha_i\,h^\beta_j.$$

Now we consider the expression for b_{ij} in (4.7). If we put

$$C_{ij} = 0,$$

we obtain from (4.7) and (2.23)

(4.14) $$b_{ij} = \beta_{ij} = \Gamma^*_{ikj}\,\partial P/\partial p_k = [ij,\ k]\,\partial P/\partial p_k - A^k_{ij}\,[kh,\ o]\,\partial P/\partial p_h.$$

We can verify that the covariant vector in T_n derived from a gradient vector $\partial\varphi/\partial\xi_\alpha + \beta_{\alpha\beta}\,\partial\varphi/\partial\pi_\beta$ in K_n is $\partial f/\partial x^i + b_{ij}\,\partial f/\partial p_j$ where $\varphi(\xi,\pi)=f(x,p)$. In fact, we have by differentiating the above identity with respect to ξ^α and π_α respectively

$$\partial\varphi/\partial\xi^\alpha = \partial f/\partial x^j\,\partial x^j/\partial\xi^\alpha + \partial f/\partial p_j\,\partial p_j/\partial\xi^\alpha,$$

$$\partial\varphi/\partial\pi_\alpha = \partial f/\partial x^j\,\partial x^j/\partial\pi_\alpha + \partial f/\partial p_j\,\partial p_j/\partial\pi_\alpha.$$

Substituting the expression $\partial\varphi/\partial\xi_\alpha + \beta_{\alpha\beta}\,\partial\varphi/\partial\pi_\beta$ for X_α in (4.2) and making use of the above equations, we obtain

$$\begin{aligned} v_i &= \left\{\frac{\partial f}{\partial x^j}\,\frac{\partial x^j}{\partial\xi^\alpha} + \frac{\partial f}{\partial p_j}\,\frac{\partial p_j}{\partial\xi^\alpha} + \beta_{\beta\alpha}\left(\frac{\partial f}{\partial x^j}\,\frac{\partial x^j}{\partial\pi_\beta} + \frac{\partial f}{\partial p_j}\,\frac{\partial p_j}{\partial\pi_\beta}\right)\right\}\left(\frac{\partial\xi^\alpha}{\partial x^i} + b_{ik}\,\frac{\partial\xi^\alpha}{\partial p_k}\right) \\ &= \frac{\partial f}{\partial x_i} + b_{ik}\,\frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial x_j}\left\{-\frac{\partial x^j}{\partial\pi_\alpha}\,\frac{\partial\pi_\alpha}{\partial x_i} + b_{ik}\,\frac{\partial\pi_\alpha}{\partial p_j}\,\frac{\partial\pi_\alpha}{\partial p_k} + \frac{\partial x^j}{\partial\pi_\beta}\left(\frac{\partial\pi_\beta}{\partial x^i} + b_{ik}\,\frac{\partial\pi_\beta}{\partial p_k}\right)\right\} \\ &\quad + \frac{\partial f}{\partial p_j}\left\{-\frac{\partial\pi_\alpha}{\partial x^i}\,\frac{\partial p_j}{\partial x^\alpha} - b_{ik}\,\frac{\partial p_j}{\partial\pi_\alpha}\,\frac{\partial\pi_\alpha}{\partial p_k} + \frac{\partial p_j}{\partial\pi_\beta}\left(\frac{\partial\pi_\beta}{\partial x^i} + b_{ik}\,\frac{\partial\pi_\beta}{\partial p_k}\right)\right\}. \end{aligned}$$

But the last two terms of the last member vanish identically, because the relations

$$\partial\pi_\alpha/\partial p_j\,\partial\xi^\alpha/\partial p_k + \partial x^j/\partial\pi_\beta\,\partial\pi_\beta/\partial p_k =$$

$$-\partial x^j/\partial\xi^\alpha\,\partial x^k/\partial\pi_\alpha + \partial x^j/\partial\pi_\alpha\,\partial x^k/\partial\xi^\alpha = 0,$$

$$-\partial\pi^\alpha/\partial x^j\,\partial p_i/\partial\pi_\alpha + \partial p_j/\partial\pi_\alpha\,\partial\pi_\alpha/\partial x^i =$$

$$\partial p_j/\partial\xi^\alpha\,\partial p_i/\partial\pi_\alpha - \partial p_j/\partial\pi_\alpha\,\partial p_i/\partial\xi^\alpha = 0,$$

hold, so that we have

$$v_i = \partial f/\partial x^i + b_{ij}\,\partial f/\partial p_j.$$

5. **Contact frame.** Differential forms $d\pi_\alpha - \beta_{\alpha\gamma}\,d\xi^\gamma$ and $dp_i - b_{ij}\,dx^j$ are the

components of a covariant vector in respective spaces K_n and T_n. In fact we have

$$d\pi_\alpha - \beta_{\alpha\gamma}\, d\xi^\gamma = \left(\partial\pi_\alpha/\partial x^i - \beta_{\alpha\gamma}\,\partial\xi^\gamma/\partial x^i\right) dx^i$$

$$+ \left(\partial x^i/\partial\xi^\alpha - \beta_{\alpha\gamma}\,\partial\xi^\gamma/\partial p_i\right) dp_i = f_\alpha^i\left(dp_i - b_{ij}\, dx^j\right)$$

by making use of (4.5) and (1.8). Hence we put

$$\delta\pi_\alpha = d\pi_\alpha - \beta_{\alpha\gamma}\, d\xi^\gamma . \tag{5.1}$$

While differential $d\xi^\alpha$ and dx^i are transformed in the manner:

$$f_\alpha^i d\xi^\alpha = dx^i - \partial x^i/\partial\pi_\alpha\, \delta\pi_\alpha . \tag{5.2}$$

We introduce here a "**contact frame**" [3][4] defined by the functions $\Gamma^{\alpha\beta}(\xi, \pi)$, homogeneous of degree -1 in the π's and symmetric in the superier indices, which are transformed in the following manner:

$$-\partial x^j/\partial\pi_\alpha + \Gamma^{\alpha\beta} f_\beta^i = \Gamma^{ij} h_i^\alpha . \tag{5.3}$$

Then the quantities defined by

$$\delta\xi^\alpha = d\xi^\alpha + \Gamma^{\alpha\beta}\,\delta\pi^\beta = d\xi^\alpha + \Gamma^{\alpha\gamma}\, d\pi_\gamma - \Gamma^{\alpha\gamma}\,\beta_{\gamma\sigma}\, d\xi^\sigma , \tag{5.4}$$

are transformed as follows

$$\delta\xi^\alpha\, f_\alpha^i = \delta x^i , \tag{5.5}$$

where

$$\delta x^i = dx^i + \Gamma^{ij}\,\delta p_j = dx^i + \Gamma^{ij}\, dp_j - \Gamma^{ij}\, b_{jk}\, dx^k .$$

In our theory, these Pfaffian forms (5.1) and (5.4) play the role of differentials of ordinary coordinates. If we resolve (5.1) and (5.4) with respect to $d\xi^\alpha$ and $d\pi_\alpha$, we obtain

$$\begin{cases} d\xi^\alpha = \delta\xi^\alpha - \Gamma^{\alpha\gamma}\,\delta\pi^\gamma , \\ d\pi_\alpha = \delta\pi_\alpha + \beta_{\alpha\gamma}\,\delta\xi_\gamma - \beta_{\alpha\gamma}\,\Gamma^{\gamma\sigma}\,\delta\pi_\sigma . \end{cases} \tag{5.6}$$

Now we can define the covariant derivatives of a vector as follows

$$DX_\alpha = dX^\alpha + C_\beta^{\alpha\gamma}\, X^\beta\, d\pi_\gamma + \Gamma_{\beta\gamma}^{\alpha} X^\beta\, d\xi^\gamma$$

$$= \nabla_\sigma X^\alpha\,\delta\xi^\sigma + \nabla^\sigma X^\alpha\,\delta\pi_\sigma , \tag{5.7}$$

where we put

$$\nabla_\sigma X^\alpha = \partial X^\alpha/\partial\xi^\sigma + \Gamma_{\beta\sigma}^{\alpha} X^\beta + \beta_{\gamma\sigma}\left(\partial X^\alpha/\partial\pi_\gamma + C_\beta^{\alpha\gamma}\, X^\beta\right), \tag{5.8}$$

$$\nabla^\sigma X^\alpha = \partial X^\alpha/\partial\pi_\sigma + C_\beta^{\alpha\sigma} X^\beta - \Gamma_{\gamma\sigma}\left(\partial X^\alpha/\partial\xi^\gamma\right.$$

$$\left. + \Gamma_{\beta\gamma}^{\alpha} X^\beta\right) - \beta_{\gamma\varepsilon}\,\Gamma^{\varepsilon\sigma}\left(\partial X^\alpha/\partial\pi_\gamma + C_\beta^{\alpha\gamma}\, X^\beta\right). \tag{5.9}$$

In particular for π_α, we have

$$\nabla_\sigma \pi_\alpha = 0 \qquad \textit{and} \qquad \nabla^\sigma \pi_\alpha = \delta_\alpha^\sigma .$$

Now we can immediately obtain various curvature tensors and torsion tensors in usual manner, but here we write the only following relation for a scalar function $f(\xi, \pi)$

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) f = S_{\mu\nu\tau} \nabla^\tau f + S_{\mu\nu\tau} \Gamma^{\alpha\tau} \nabla_\alpha f,$$

where

$$S_{\mu\nu\tau} = \partial_\mu \beta_{\tau\nu} / \partial \xi^\mu - \partial \beta_{\tau\mu} / \partial \xi^\nu + \beta_{\varepsilon\mu} \partial \beta_{\tau\nu} / \partial p_\varepsilon - \beta_{\varepsilon\nu} \partial \beta_{\tau\mu} / \partial p_\varepsilon . \tag{5.10}$$

6. **Absolute differential in T_n.** In the space K_n with linear connections $\left(\Gamma^{\alpha}_{\beta\gamma}, C^{\alpha\gamma}_{\beta}\right)$ the expression $DX^\alpha = dX^\alpha + \Gamma^{\alpha}_{\beta\gamma} X^\beta d\xi^\gamma + C^{\alpha\gamma}_{\beta} X^\beta d\pi^\gamma$ is an absolute differential of the vector X^α. We denote by

$$DY^i = dY^i + Y^j L^i_{jk} dp_k + Y^j D^{ik}_j dp_k \tag{6.1}$$

the absolute differential of the vector Y^i in T_n derived from X^α in K_n by equation (4.1). From $Y_i DY^i = X_\alpha DX^\alpha$, we have

$$dY^i + Y^j L^i_{jk} dx^k + Y^j D^{ik}_j dp_k = \left(dX^\alpha + \Gamma^{\alpha}_{\beta\gamma} X^\beta d\xi^\gamma + C^{\alpha\gamma}_{\beta} X^\beta d\pi_\gamma\right) f^i_\alpha .$$

Since $dX^i = df^i_\alpha X^\alpha + dX^\alpha f^i_\alpha$ for arbitrary vector, we must have

$$L^i_{jk} dx^k + D^{ik}_j dp_k = - df^i_\beta h^\beta_j + \left[\Gamma^{\alpha}_{\beta\gamma} d\xi^\gamma + C^{\alpha\gamma}_{\beta} d\pi_\gamma\right] f^i_\alpha h^\beta_j . \tag{6.2}$$

On the other hand, since π_α and p_i are covariant vector in K_n and T_n respectively, we have

$$dp_k - L^j_{ki} p_j dx^i - D^{ji}_k p_j dp_i = \left(d\pi_\alpha - \Gamma^{\beta}_{\alpha\gamma} \pi_\beta d\xi^\gamma\right) h^\alpha_k \tag{6.3}$$

$$= \delta\pi_\alpha h^\alpha_k = \delta p_k = dp_k - b_{ki} dx^i .$$

hence, we obtain

$$D^{ji}_k p_j = 0 \qquad or \qquad D_{kji} p^j = 0. \tag{6.4}$$

$$L^j_{ki} p_j = b_{ki}. \tag{6.5}$$

By covariant derivation of the fundamental tensor, we have

$$dg^{ij} + L^i_{lk} g^{lj} dx^k + L^j_{lk} g^{il} dx^k + D^{ik}_l g_{lj} dp_k$$

$$+ D^{jk}_l g^{il} dp_k = Dg^{\alpha\beta} f^i_\alpha f^j_\beta = 0.$$

Hence, we have

$$\partial g^{ij} / \partial x^k = L^{ji}_{\ k} + L^{ij}_k, \tag{6.6}$$

$$\partial g^{ij} / \partial p_k = D^{jik} + D^{ijk}. \tag{6.7}$$

From (6.4), (6.7) and relation $p_j \partial g^{ij} / \partial p_k = 0$, we have

$$D^{jik} p_j = 0. \tag{6.8}$$

By equation (6.4), (6.3) becomes

$$dp_k - L^l_{ki} p_l dx^i = (d\pi_\alpha - \beta_{\alpha\gamma} d\xi^\gamma) h^\alpha_k. \tag{6.9}$$

If we put $\delta\pi_\alpha = 0$, from (6.9) we have $dp_k = L^l_{ki}\, p_l\, dx^i$, hence (6.2) becomes

$$\left(L_j{}^i{}_k + D^{ih}_j\, L^l_{h\,k}\, p_l\right) dx^k =$$
$$- h^\beta_j \left[\left(\partial f^i_\beta/\partial \xi^\alpha + \partial f^i_\beta/\partial \pi_\gamma\, \beta_{\gamma\alpha}\right) d\xi^\alpha - \Gamma^{*\alpha}_{\beta\gamma}\, d\xi^\gamma\, f^i_\alpha\right].$$

Therefore, we have

$$L^{*l}_{jk} f^k_\alpha f^j_\beta = - \left(\partial f^i_\beta/\partial \xi^\alpha + \partial f^i_\beta/\partial \pi_\gamma\, \beta_{\gamma\alpha}\right) + \Gamma^{*\gamma}_{\beta\alpha} f^i_\gamma, \tag{6.10}$$

where

$$L^{*i}_{jk} = L^i_{jk} + D^{ih}_j\, L^l_{h\,k}\, p_l. \tag{6.11}$$

When we alternate the indices α and β in (6.10) and substruct the result from (6.10), we have

$$\left(L^{*i}_{jk} - L^{*i}_{kj}\right) f^k_\alpha f^j_\beta = - S_{\alpha\beta\tau}\, \partial x^i/\partial \pi_\tau, \tag{6.12}$$

because $\Gamma^{*\gamma}_{\beta\alpha}$ is symmetric in indices β and α.

If the torsion tensor $S_{\alpha\beta\tau}$ vanishes, from (6.12) the quantity L^{*i}_{jk} is symmetric in lower indices, i.e.

$$L^{*i}_{jk} = L^{*i}_{kj}. \tag{6.13}$$

Assume that the quantity D^{ijk} is symmetric in indices i and j, from (6.7) we have

$$D^{ijk} = \tfrac{1}{2}\, \partial g^{ij}/\partial p_k, \tag{6.14}$$

then, in such a case, the conditions (6.4) and (6.8) upon the quantities D^{ijk} are satisfied. In this way we have, from (6.11),

$$L^{*i}_{jk} = L^i_{jk} + A^{ih}_j\, L_{hok}, \tag{6.15}$$

where

$$A^{ih}_j = \sqrt{2P}\, D^{ih}_j \qquad \text{and} \qquad L_{hok} = L_{hik}\, l^i.$$

By alternating the lower indices j, k in L^{*}_{ijk} and by summing the result to L^{*}_{ijk}, we have

$$L^{*}_{jik} + L^{*}_{ijk} = L_{jik} + L_{ijk} + 2A^{\ h}_{ij}\, L_{hok} = \partial g^{ij}/\partial x^k + 2A^{\ h}_{ij}\, L_{hok},$$

because

$$Dg_{ij} = dg_{ij} - (L_{ijk} + L_{jik})\, dx^k - 2C^{\ k}_{ij}\, dp_k = Dg_{\alpha\beta}\, h^\alpha_i\, h^\beta_j = 0.$$

Hence we have

$$\partial g_{ij}/\partial x^k = L_{ijk} + L_{jik},$$

accordingly from (2.13), (2.16) and (2.17), we conclude

$$L^i_{jk} = \Gamma^i_{jk}. \tag{6.16}$$

In this case the transformed space T_n is also the space K_n with the connections C's and Γ's.

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ISOLATION OF AROMATIC HYDROCARBONS FROM PETROLEUM BY AZEOTROPIC AND EXTRACIVE DISTILLATION AND BY "AZEODISTEX" METHOD

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As the physical methods for separating aromatic hydrocarbons from petroleum the extraction with solvent [1], adsorption with silica gel [2] and crystallization by refrigiration [3] have been hitherto known. By the solvent extraction method only extreme selective solvents, such as sulfur dioxide and aniline, are effective in concentrating the aromatics, but it is difficult to isolate aromatics in pure state from the oil of high content of aromatics. The author has formerly tried to separate aromatics by extraction with ethyl alcohol and found to have no effect. The adsorption method is only available when a small content of aromatics is to be removed or recovered, and freezing method is effective to isolate benzene from a benzene fraction containing more than 60% or to isolate p-xylene from a sufficiently concentrated xylene fraction, and therefore they are not general methods.

When the petroleum is to be separated by the fractional distillation, benzene and the other aromatic hydrocarbons distill over at wide boilng range accompanied with another type of hydrocarbons, with which they form complex azeotropic mixtures [4], and therefore it is almost impossible to isolate pure aromatics completely from petroleum only by fractionation or rectification. In this case it is very effective for separation to distill adding another substance which forms a new azeotrope with original hydrocarbon but is readiy removed by washing with water. Rossini, Mair and coworkers [5] have applied this azeotropic distillation in their researches of constituents of Mid-continent petroleum and could isolate aromatic hydrocarbons from high boiling gasoline fraction using acetic acid etc. Recently it has been reported that the azeotropic distillation is able to be applied effectively in the puirfication of commercial benzene [6] and separation or recovery of benzene and toluene [7].

The author has isolated the aromatic hydrocarbons such as benzene, toluene, xylene and mesitylene by the azeotropic distillation from the gasolines produced in Formosa and found the possibility of industrial realization [8]. As the azeotropic agent methanol and ethanol were used in different ratio to the oil to compare the effect and yield.

As a method for separating the aromatics, the author tried also the distillation in the presence of solvent of small volatility which has greater affinity with aromatics than with non-aromatics. These additives would effect on the relative volatility of aromatic and non-aromatic constituents and decrease the composition of aromatics in the equilibrium vapor phase and hence increasing the content of non-aromatics in the forerunning distillate. In practice by our experiments, however, when the distillation is carried out adding this solvent

alone or further being added with an azeotropic agent besides this solvent, the effect of separation increases in small extent in comparision with that without this solvent and has been found to have no practical value.

Recently Griswold and coworkers [9] have tried successfully an extractive distillation, so-called distex process, to isolate pure hydrocarbons from petroleum, and developed a new field in the method of separation by solvent. They carried out this procedure with a contineous distillation and their laboratorial test was performed by a simi-industrial pilot plant. Afterward this extractive distillation was also applied for the isolation of C4 hydrocarbons [10].

The author modified Griswold's process and tested using a glass batch fractionating apparatus provided with a packed column, the raw material being charged every batch in the still, while the extractive agent being dropped contineously from a dropping funnel of the stillhead onto the top of the column to submit to the extractive distillation.

The author, now, jointed this extractive distillation with the azeotropic distillation, which were carried out simultaneously in the same column, and found a great effect for separation. The author want to designate this new process as an "Azeodistex". To the methods of separation by solvent, of which extraction, azeotropic distillation and Griswold's extractive distillation or distex, were known, we could add our azeodistex process, which has great effect for separating each type of hydrocarbons from the complex mixture such as petroleum. This process is, however, available only when the compounds of different types (e. g. paraffins, naphthenes and aromatics) are to be separated from the fraction of narrow boiling range including the almost same size of molecules. Therefore, before this process is applied, the mixture should be fractionated to obtain a substantially constant boiling fraction by the rectifying apparatus of high efficiency. It is desirable that the product from azeodistex is again submitted to the rectifying distillation.

Experimental

Raw Materials

The raw materials used in these experiments were aromatic fractions obtained by twice fractionation of the gasolines produced in Shukkoko and Kinsui (Shinchiku, Formosa) using the fractionating apparatus provided with a packed column of 3 meters height and a stillhead [8]. The characteristics of these raw materials are shown in TABLE I.

TABLE I

Characteristics of Raw Materials

No.	Name of Materials	Boiling Range °C	n_D^{20}	Aromatic Content
1	Benzene fraction from Shukkoko	75-81	1.4578	62%
2	Benzene fraction from Kinsui	75-80	1.4469	44%

3	Toluene	fraction	from	Shukkoko	110-112	1.4716	77%
4	Toluene	fraction	from	Kinsui	108-112	1.4617	68%
5	Xylene	fraction	from	Shukkoko	137-138	1.4770	81%
6	Xylene	fraction	from	Kinsui	137-138	1.4727	72%
7	Mesitylene	fraction	from	Shukkoko	160-165	1.4647	60%

Apparatus and Procedure

The apparatus used for azeotropic distillation was a glass distilling flask of 500 cc. capacity connected to a glass column of 120 cm height packed with small glass raschig rings and provided with a stillhead. It was estimated to be equivalent to 6 theoretical plates. The apparatus used for distex (extractive distillation) and azeodistex was quite similar one except that the stillhead was provided with a dropping funnel for feeding the extractive agent.

Azeotropic distillation was carried out by the batch process after charging a material oil and an azeotropic agent of definite ratio into the still, while in the case of distex the extractive agent was contineously charged at constant velocity from the dropping funnel onto the top of column during every batch distillation.

It the both cases reflux ratio was maintained at about 6 to 1, and the distillation velocity at 40 cc per hour. The distillates were cut into the same volume fractions, which were separately washed with water to remove the azeotropic solvent and the physical constant of recovered oils was determined to know the effect of separation. These fractions were combined, based on the refractive indices, to three portions designated as non-aromatic, intermediate, and aromatic and the yields of each portion were obtained to compare the separating effect. The residue of distex and azeodistex consists of only extractive agent, which can be directly or after once rectification used for next process. Examples for azeotropic and extractive distillations and azeodistexes were shown in TABLE II, III and IV respectively.

TABLE II

One Example for Azeotropic Distillation

Material oil : No. 3 Toluene fraction (110-112) 150 cc
Azeotropic agent : Methyl alcohol 200 cc } 350 cc

Fraction No.	Boiling point	Distillate cc.	Recovered oil cc.	Refractive indices	Yields of comb. oil
1	62.5-64.0	2 0*	8. 7	1. 4258	
2	64.0-64.2	2 0*	7. 5	1. 4348	
3	64.2-64.4	2 0	7. 5	1. 4407	
4	64.4-64.5	2 0	7. 0	1. 4449	Non-arom.
5	64.5-64.5	2 0	7. 5	1. 4488	65. 2 cc
6	64.5-64.5	2 0	7. 5	1. 4556	(44.3%)
7	64.5-64.7	2 0	6. 5	1. 4605	
8	64.7-65.0	2 0	6. 5	1. 4658	
9	65.0-65.3	2 0	6. 5	1. 4705	

10	65.3-65.5	20	7. 0	1. 4772	Intermediate mixture 26.3 cc (17.9%)
11	65.5-65.5	20	6. 5	1. 4823	
12	65.5-65.7	20	6. 5	1. 4843	
13	65.7-66.0	20	6. 3	1. 4870	
14	66.0-67.5	20	7. 5	1. 4906	Aromatic 55.5 cc (37.8%)
15	67.5-74.0	24	10. 0	1. 4919	
16	Residue	40	38. 0	1. 4934	

* These fractions were consisting of two layers.

TABLE III

One Example for Extractive Distillation

Material oil : No. 3 Toluenefraction (110-112) 150 cc
Extractive agent : Furfural 300 cc
} 450 cc

Fraction No.	Boiling point	Distillate cc.	Refractive indices	Yelids of combined oil
1	109.0-111.0	10	1. 4446	Non-aromatic 80.0 cc (53.3%)
2	111.0-111.3	10	1. 4438	
3	111.3-111.5	10	1. 4516	
4	111.5-111.5	10	1. 4686	
5	111.5-111.6	10	1. 4681	
6	111.6-111.8	10	1. 4607	
7	111.8-112.0	10	1. 4589	
8	112.0-112.5	10	1. 4620	
9	112.5-112.5	10	1. 4712	Intermediate mix. 30.0 cc (20.1%)
10	112.5-112.5	10	1. 4815	
11	112.5-112.5	10	1. 4866	
12	112.5-112.5	10	1. 4890	Aromatic 40.0 cc (26.6%)
13	112.5-112.8	10	1. 4906	
14	112.8-113.0	10	1. 4912	
15	113.0-113.0	10	1. 4917	
16	Residue	290	—	Furfural

TABLE IV

One Example for Azeodistex Process

Material oil : No. 3 Toluene fraction 100 cc
Azeotropic agent : Ethyl alcohol 100 cc
Extractive agent : Furfural 200 cc
} 200 cc

Fraction No.	Boiling point	Distillate cc.	Recovered oil cc.	Refractive indices	Yields of comb. oil
1	74.0-75.0	15	5. 0	1.4215	Non-aromatic 31.0 cc (33.0%)
2	75.0-75.2	15	1. 5	1.4299	
3	75.2-75.4	15	1. 0	1.4267	
4	75.4-75.5	15	2. 0	1.4182	
5	75.5-75.5	15	3. 5	1.4136	
6	75.5-76.0	15	5. 5	1.4122	
7	76.0-77.5	15	5. 5	1.4293	
8	77.5-78.0	15	3. 5	1.4527	Intermediate 5.5 cc (5.8%)
9	78.0-78.0	15	3. 5	1.4532	
10	78.0-94.5	15	5. 5	1.4736	
11	94.5-106.0*	15	11. 5	1.4909	Aromatic 57.5 cc (61.2%)
12	106.0-109.5*	15	14. 0	1.4942	
13	109.5-110.0*	15	15. 0	1.4953	
14	110.0-110.0	17	17. 0	1.4953	
15	Residue	196			Furfural

* The elevation of boiling point was due to the shortage of azeotropic agent.

Experimental Results

Although the experimental results of azeotropic distillation, carried out with benzene, toluene, xylene and mesitylene fractions from Shukkoko and Kinsui gasolines using different ratio of azeotropic agents such as ethanol and methanol, have been reported [8], these data were here again recorded so as to compare with those of processes described below.

The author used the same raw materials and subjected to distex (extractive distillation) and azeodistex using furfural and aniline as extractive agents, and ethanol and metnanol as azeotropic agents. The results were shown in TABLE V, VI and VII.

TABLE V

Results of Separation with Benzene Fraction

Material Oil	Azeotropic agent	Extractive agent	Ratio of Oil: Az.: Ex. Ag.	Nonaromatic %	Intermediate %	Aromatic %
No. 1	Ethanol	Without	2 : 1	82.9	17.1	0
Benzene	do.	do.	1.5: 1	81.5	18.5	0
Fraction	do.	do.	1 : 1	73.8	16.2	9.9
from	do.	do.	1 : 1.5	79.5	19.5	1.0
Shukkoko	Methanol	Without	1 : 1	65.7	6.0	28.3
	Without	Furfural	1 : 2	73.7	10.5	15.8
	Ethanol	Furfural	1:1:2	59.6	9.4	31.0
	do.	do.	do.	59.8	8.2	32.0
No. 2	Ethanol	Without	1 : 1	81.5	12.3	6.2
Bz. Fr.	do.	do.	1.5: 1	79.9	10.2	9.9
from	Methanol	do.	1 : 1	71.0	9.7	19.3
Kinsui	Ethanol	Furfural	1:1:2	63.8	10.0	26.2

TABLE VI

Results of Separation with Toluene Fraction

Material Oil	Azeotropic Agent	Extractive Agent	Ratio of Oil: Az.: Ex. Ag.	Nonaromatic %	Intermediate %	Aromatic %
No. 3	Without	Without	—	18.9	81.1	0
Toluene	Ethanol	Without	1 : 1	49.3	20.1	30.6
Fraction	do.	do.	1.5: 2	51.1	16.9	32.0
from	do.	do.	1 : 2	51.7	13.5	34.8
Shukkoko	Methanol	Without	1.5: 2	44.3	17.9	37.8
	Without	Furfural	1 : 1	53.3	46.7	0
	do.	do.	1 : 2	53.3	20.1	26.6
	Ethanol	Furfural	1:1$\frac{1}{2}$:2	48.9	17.4	33.7
	do.	do.	1:1: 2	33.0	5.8	61.2
	Methanol	do.	1:2: 2	35.7	13.0	51.3
	Ethanol	Aniline	1:1: 1$\frac{1}{2}$	46.5	30.8	22.7
No. 4	Without	Without	—	52.4	47.6	0
Toluene	Ethanol	Without	1: 2	53.4	23.6	23.0
Fraction	do.	do.	1: 1.5	62.3	14.3	23.4
from	do.	do.	1: 1	57.2	22.8	20.0
Kinsui	Methanol	do.	1: 1.5	47.6	18.4	34.0
	do.	do.	1: 2	55.1	12.3	32.6

Without	Furfural	1: 2	62.4	18.8	18.8
Without	Aniline	1: 2	50.0	37.5	12.5
Ethanol	Furfural	1:1: 2	39.3	10.7	50.0
Methanol	Furfural	1:1$\frac{1}{2}$:2	46.8	12.2	41.0
do.	do.	1:2: 2	58.3	16.3	25.5
Ethanol	Aniline	1:1: 2	60.0	12.8	27.2

TABLE VII

Results of separation with

Xylene and Mesitylene Fractions

Material Oil	Azeotropic Agent	Extractive Agent	Ratio of Oil:Az.: Ex. Ag.	Non-Aromatic %	Interm-ediate %	Aromatic %
No. 5	Without	Without	—	6.8	92.8	0
Xylene	Ethanol	Without	1: 1.5	34.3	15.0	50.7
Fraction	do.	do.	1: 2	34.7	15.4	49.9
from	Methanol	do.	1: 2	26.7	13.0	60.3
Shukkoko	Acetic acid	do.	1: 1.5	30.9	16.3	52.8
	Without	Furfural	3: 4	20.8	30.1	48.2
	Ethanol	Furfural	1: 1$\frac{1}{2}$:2	23.1	10.6	65.3
No. 6						
Xy. Fr.	Ethanol	Without	1: 2	47.7	13.3	39.0
Kinsui	do.	do.	1: 1.5	52.1	9.1	38.8
No. 7	Ethanol	Wthout	1: 4.5	44.2	55.8	0
Mesitylene	do.	do.	1: 6	40.0	0	60.0
Fraction from	Acetic acid	do.	3: 8	58.1	16.3	25.6
Shukkoko	Butylic acid	do.	1: 1	65.3	24.0	10.7
	Aniline	do.	1: 1	60.2	11.9	27.9

As shown from the tables, the separating effect by the distex is not so remarkable, but it can be greatly increased by the azeodistex using azeotropic agent besides extractive solvent. It would be worthy for attention that by the azeotropic distillation using simple solvent, methanol is more effective for separation than ethanol, but by the azeodistex, on the other hand, ethanol has greater effect than methanol. It may be owing to the influence of extractive agent on the equilibrium composition of azeotropic solvent and hydrocarbon oil. As an extractive agent furfural was more effective than aniline in the azeodistex process, although the aniline itself has a greater selectivity and affinity to the aromatic compounds than furfural does as shown by the results of extraction.

Now, the author tried the distillation in the presence of additive such as furfural and also the azeotropic distillation adding an azeotropic agent besides furfural. The separation by this method was effective in some extent but was not so remarkable than by azeodistex process, although the experimental results were here omitted.

The characteristics of the aromatic hydrocarbons obtained by the azeotropic and extractive distillations from the gasolines are shown in TABLE VIII.

TABLE VIII

Characteristics of Aromatic Hydrocarbons Obtained from the Gasolines

I. Benzene

Boiling point : 79—80° C

Azeotropic boiling point with ethanol 68° C

with methanol 58° C

Refractive index : $n_D^{27} = 1.4906$; $n_D^{20} = 1.5004$

Specific gravity : $d_4^{27} = 0.8667$; $d_4^{20} = 0.8741$

Distillation analysis (by precise rectifying apparatus)

B. P. C	Distillate	n_D^{20}
77.0—79.8	20 %	1.4982
79.8—80.0	20	1.5003
80.0—80.0	20	1.5003
80.0—80.0	20	1.5007
80.0—81.0	20	1.5015

II. Toluene

Boilimg point : 110-111° C

Azeotropic boiling point with ethanol 77° C

Refractive inbex : $n_D^{30} = 1.4884$; $n_D^{20} = 1.4944$

Specific gravity : $d_4^{30} = 0.8592$; $d_4^{20} = 0.8595$

Purity : 97.3%

Distillation analysis

B. P. C	Distillate	n_D^{20}
110—110.5	14 %	1.4927
110.5	14	1.4944
110.6	14	1.4947
110.6	14	1.4951
110.6	14	1.4952
111.0	14	1.4953
Residue	14	1.4929

III. Xylene

Boiling point : 137—139°

Refractive index : $n_D^{30} = 1.4888$; $n_D^{20} = 1.4935$

Specific gravity : $d_4^{30} = 0.8598$; $d_4^{20} = 0.8594$

Purity : 95.0% chiefly consist of m-xylene

Distillation analysis

B.P.	Distillate	n_D^{20}
137—138	8.3 %	1.4907

138—136	84.3	1.4944
Residue	9.8	1.4202

IV. Mesitylene

Boiling point : 162—164 C

Refractive index : $n_D^{30} = 1.4905$; $n_D^{20} = 1.4945$

Specific gravity : $d_4^{30} = 0.8518$; $d_4^{20} = 0.8598$

Purity : 97.5 %

Distillation analysis

B.P. C	Distiliate	n_D^{20}
155.0—161.0	1 6 %	1.4913
161.0—162.5	1 6	1.4939
162.5—162.5	1 6	1.4947
162.5—163.0	1 6	1.4949
163.0—163.5	1 6	1.4954
Residue	1 6	~

Conclusion

The separating effects by the azeotropic and extractive distillations and by azeodistex were compared by the yields of pure aromatic hydrocarbons obtained, and the results were summarized as follows:

When the benzene fraction of Shukkoko (aromatic content 62%) was used as rawmaterial, the yield of pure benzene was about 10% by the azeotropic distillation with ethanol, 28% with methanol, 16% by distex with furfural, and 32% by azeodistex with ethanol-furfural. With the benzene fraction from Kinsui (aromatic content 44%) 10% by azeotropic distillation with ethanol, 26% by azeodistex with ethanol-furfural were obtained. Was the toluene fraction from Shukkoko (aromatic content 77%) used, the yield was 34% by the azeotropic distillation with methanol, 26% by the distex with furfural, 51% by the azeodistex with methanol-furfural, 61% with ethanol-furfural, 23% with ethanol-aniline, while with to luene fraction from Kinsui (aromatic content 68%) the yield was 23% by azeotropic distillation with ethanol, 34% with methanol, 19% by distex with furfural, 41% by azeodistex with methanol-furfural, 50% with ethanol-furfural, and 27% with ethanol-aniline. From the xylene fraction (aromatic content 81%) 50% by azeotropic distillation with ethanol, 60% with methanol, 65% by azeodistex with ethanol-furfural were obtained. From the mesitylene fraction (aromatic content 60%) pure mesitylene could be isolated almost quatitatively (60%) by only azeotropic distillation with ethanol when this is used in excess.

From these results it is shown that the most effective separation of aromatic compounds from a petroleum can be achieved by the azeodistex process using ethanol as azeotropic agent and furfural as extractive agent. When the separation is to be carried out merely by the azeotropic distillation, methanol is most effective as azeotropic agent, although with higher

boiling fractions ethanol can be used with sufficient result.

The aromatic hydrocarbons thus obtained have purity of more than 90%, which can be increased by the rectification followed by further azeotropic distillation or by repeating these processings, although the yield decreases remarkably.

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ON THE SURFACE FIGURE OF THE FARTH, THE MOON AND MARS (Report I)

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Part I. Surface figure of the earth

That the form of the surface figure of the earth is relatively regular is generally accepted by many authors, among them Love's harmonic analysis [1] of the continental distribution and Wegener's floating theory of continents [2] are the most interesting ones on the physical point of views. These two works are, however, seems to be quite different in a very important point. Love's analysis implicitly shows that the present distribution of continents is stable one, at least quasi-stable one, while according to Wegener's theory it is not. Latter shows that continents are moving separately under the actions of gravitational field and that of viscosity of underlaying strata of sima.

Wegener's theory is very ingeneous but rather qualitative with many weak points. It gives no explanation of the distribution and form of fissures separating different continents. On the otherside, Love's analysis implies that there is some physical meaning to make the continents to take the present forms. But it is very obvious that any continuous function distributed on a spherical surface can be expressed by a series of spherical harmonics, with an accuracy depending on the number of terms taken. It is, therefore, far too hasty to conclude that there is some physical meaning in the present forms of the earth's surface.

If however, the harmonic series fits on some part of the figure – more than one-half of the whole surface – also fits on the remaining part, the physical significance may be increased. Of course, the series thus obtained cannot be expected to fit on the remaining part with the same accuracy as that obtained by using the whole data on the surface.

As an example, the present author tried to test in what measure the series obtained by using the data of the old continents (0°–210° E. longitude) can express the figure of the new continent (0°–150° W. longitude).

Chapter I. Harmonic series representing the figure of old continents

Love took the height (h) + 1 for continental area, – 1 for oceanic one and 0 for coastal and shallow sea region. As he has done, spherical harmonic terms up to the third order, which contains 16 constants, are taken and those constants are determined by the method of the least squares using data in the region 0° – 210° E (150° W) of those given by Love in his paper with the addition of data of the Arctic sea and Antarctic Continent.

The result is

$$h = 0.1695 + 0.860\,P_1(\mu) + (0.206\sin\psi - 0.320\cos\psi)\,P_1'(\mu) + 0.0738P_2(\mu) + (0.3881\cos\psi - 0.2654\sin\psi)\,P_2^1(\varphi) + (-0.0487\cos2\psi + 0.665\sin2\psi)\,P_2^2(\mu) + 0.08883P_3(\mu) + (-0.6552\cos\psi + 0.05778\sin\psi)\,P_3'(\mu) + (-0.09715\cos2\psi + 0.1426\sin2\psi)\,P_3^2(\mu) + (0.0494\cos3\psi + (0.0494\cos3\psi + 0.0180\sin3\psi)\,P_3^3(\mu),$$

where $P_1(\mu) = \mu$, $P_1' = \sin\theta \dfrac{dP_1}{d\mu}$, $P_2(\mu) = \dfrac{1}{2}(3\mu^2 - 1)$, $P_2'(\mu) = \sin\theta \dfrac{dP_2}{d\mu}$,

$P_2^2(\mu) = \sin^2\theta \dfrac{d^2P_2}{d\mu^2}$, $P_3(\mu) = \dfrac{1}{2}(5\mu^3 - 3\mu)$, $P_3'(\mu) = \sin\theta \dfrac{dP_3}{d\mu}$, $P_3^2(\mu) = \sin^2\theta \dfrac{d^2P_3}{d\mu^2}$,

$P_3^3(\mu) = \sin^3\theta \dfrac{d^3P_3}{d\mu^3}$,

and $\mu = \cos\theta$, and μ is the north polar distance. Fig. 1 shows the distribution of h on the whole earth. It gives a large continent in the area between the longitudes 200° E and 300°

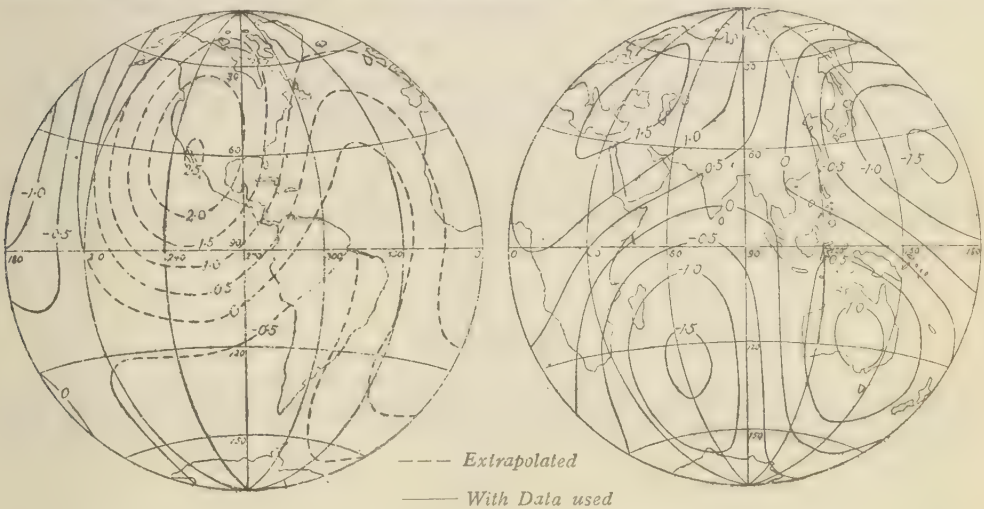

Fig. 1

E (or 160° W to 60° W) in the northern hemisphere. This nearly coincides wth the new continent (North America). South America is not represented by this equation.

But this simple calculation seems to give a strong support for that Love's harmonic expression of the surface figure of the earth is physically important. On the contrary, it does not favour to Wegener's floating theory without critisism. If any continental movement in Wegener's theory really takes place, continents are considered to move in one of alternative modes. The first is the movement as a whole from east to west without any appreciable relative displacement, since the time when the present form of continental mass were first formed.

The second is relative movement due to different modes of oscillations of the earth. The continental distribution given by equation (1) make us to attribute it to some oscillatory movement of the earth. If such a movement is what took place at some epoch in the past and at present entirely ceased, the first mode of continental movement must take place. If, on the contrary, the earth is oscillating even at present, the second mode of movement also can be expected. But this second mode of movement is less provable, if not impossible, as such an oscillation must have a very short period, and it cannot stimulate a gradual motion of continents as pointed out by Wegener.

There is the third alternative. It is to assume that the continental distribution represented by equation (1) is a stable form of the earth's surface figure and the present surface figure is not. Under this assumption, continents has a tendency to take its stable position. Westward movement of American continents may be explained as one of such movement.

The fourth altenative is that when the continents are driven westwards from their stable position given by equation (1) by gravitational field, American continents are being left behind owing to large frictional force due to the viscossity of sima.

After this idea, American continents must be approaching to Africa, which is quite contrary to Wegener's conclusion. But on these last two alternatives further discussions are necessary.

From these results it must be concluded that Love's analytical expression have some physical meaning. If it is so, Wegener's theory must have some amendments to be reconcile with Love's analysis.

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SOME STUDIES ON VOLCANO ASO AND KUJIU (PART I)

ANNUAL VOLCANIC VARIATON OF KUJIU

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Abstract

As the numerical data for the study on volcanic action, the writers measured the amount of spouted sulphur of Kujiu. Comparing the observed results with the change of climatic elements during the same time, they discussed the correlation between the annual volcanic variation and the change of climatic elements.

1. Introduction

Volcano Kujiu in Ôita prefecture is 1,300m above the sea-level and has active craters. Here sulphur has been produced by using draught-method since 1896. During 13 monthes from May 1947 to May 1948, the writers have measured the yield of spouted sulphur of Kujiu which are shown in table 1, for the studies on annual volcanic variation. The present paper deals with the observed results above mentioned.

TABLE 1.

Average monthly elements at Kujiu sulphur mine

Month	Spouted sulphur	Atmos. Press.	Atmos. Temp.	Precipit.
(1947)	ton	+600mm	°C	mm
5	33.170	74.83	12.58	173.7
6	20.700	73.45	15.81	491.2
7	21.700	75.64	21.95	212.8
8	31.355	78.04	21.55	120.6
9	41.225	75.96	18.46	59.0
10	89.110	78.53	10.18	47.0
11	95.690	79.93	6.30	26.6
12	129.435	78.61	1.17	164.0
(1948)				
1	111.995	77.17	0.25	49.0
2	70.835	76.73	1.89	80.0
3	69.955	77.36	4.36	131.8
4	49.950	76.22	10.50	105.1
5	56.955	75.70	13.76	152.9
Total	816.075	998.17	138.76	1724.3
Mean	62.775	76.781	10.674	132.64

Alld data reduced refer to May

Month	Spouted sulphur	Atmos. Press.	Atmos. Temp.	Precipit.
	ton	mm	°C	mm
1	66.0	1.80	−13.50	−110.0
2	20.0	1.30	−11.50	− 78.0
3	17.0	1.90	− 9.50	− 24.0
4	− 5.0	0.60	− 3.00	− 49.0
5	0.0	0.00	0.00	0.0
6	−13.0	−1.30	3.00	229.0
7	−14.0	0.70	9.00	43.0
8	− 8.0	3.00	9.00	− 47.0
9	0.0	0.90	6.00	−105.0
10	46.0	3.40	− 3.00	−117.0
11	51.0	4.30	− 7.50	−136.0
12	73.0	3.30	−12.50	3.0
Mean	19.42	1.66	− 2.79	−32.58

2. Arrangement of Observed Values

Let S (ton): the spouted sulphur
P (mm Hg): the atmospheric pressure
T (deg. Cent.): the atmospheric temperature
W (mm): the precipitation
and express their monthly amount in the following equation;

$$N=N_0+N_1 \cos(t-\theta_1)+N_2 \cos(2t-\theta_2)+\cdots\cdots$$

then we have the results shown in table 2.

TABLE 2.

		N	N_1	θ_1	N_2	θ_2	N_3	θ_3	N_4	θ_4	N_5	θ_5	N_6	θ_6	Pob. error.
Spouted sulphur	Ston	19.42	39.87	333°	12.04	280°	4.47	64°	4.33	332°	8.81	287°	0.58	0°	±5%
Atmos. Press.	Pmm	1.66	1.80	305°	0.34	160°	0.36	295°	0.51	91°	0.56	291°	0.22	180°	±5%
Atmos. Temp.	T C	−2.79	10.97	188°	1.56	84°	0.52	81°	0.46	69°	0.91	123°	0.21	0°	±3%
Preciitp.	Wmm	32.58	89.35	139°	69.35	313°	16.72	193°	58.48	244°	7.94	332°	22.75	0°	±30%

It will be seen from the results that the annual variation term and the semi-annual variation term are important. And errors are as following;

$$S=19.42+39.87 \cos(t-333)+12.04 \cos(2t-280) \pm 20\%$$

$$P=1.66+1.80 \cos(t-305)+0.34 \cos(2t-160) \pm 16\%$$

$$T=-2.79+10.97 \cos(t-188) \pm 7.6\%$$

$$W=W_0+89.35\cos(t-139)+69.35 \cos(2t-313) \pm 49\%$$

Therefore, the error of precipitation is pretty large, and the analysis of the other quantities are moderately perpect (Fig. 1, 2, 3). Now we will investgate the first two terms.

1. Spouted sulphur

Fig. 1 shows that the annual variation term is greater than the semi-annual variation term. The former attains the maximum in December and the minimum in June. The resultant of annual and semi-annual variation term attains the maximum in December and the minimum in July, in other words the maximum in winter and the minimum in summer. Therfore we can recognise that the volcanic activity of Kujiu is greater in winter and less in summer analogous to other volcanoes.

2. Spouted sulphur and the variation of atmospheric temperature

Fig. 1, 2 show that lag in phase angle between the temperature variation and the spouted sulphur exists, and it indicates that the temperature is less effective on S compared with the other elements.

The analysis of volcanic gas of Kujiu shows that it contains only 0.0012−0.0062% sulphur[1], therfore we may put the sulphur contained in spouted volcanic gas out of question. Moreover, as the temperature of mouth of draught is 125°−130° C. the annual variation

of atmospheric temperature can not be effective on the sublimation of sulphur.

According to the fact above descrived, we may neglect the effect of temperature variation on the annual variation of spouted sulphur.

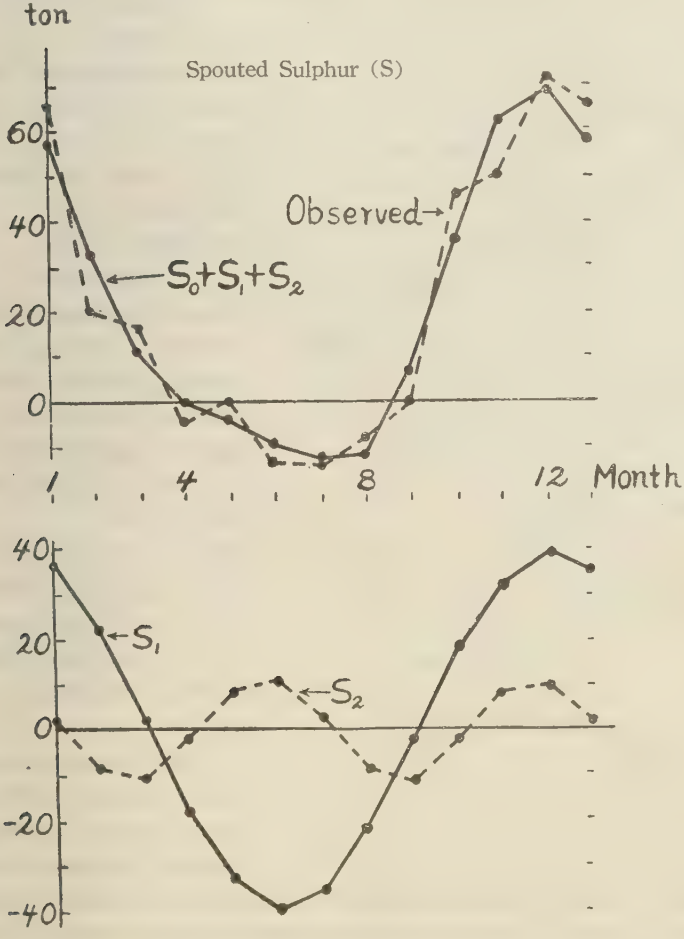

Fig. 1

3. The variation of atmospheric pressure and that of precipitation

Fig. 2, 3 show that t term and 2t term of the pressure and the precipitation change simultaneously, and the decrease of atmospheric pressure corresponds to the increase of precipitation. On the volcanic activity the precipitation should have the influence similar to that of atmospheric pressure, and it is easily seen that the resultant of two quantities takes effect upon the volcanic activity.

4. The variation of atmospheric pressure and that of spouted sulphur

As above mentioned the variation of spouted sulphur or the volcanic activity of Kujiu depends chiefly upon the variation of atmospheric pressure. Both Fig. 1, 2 denote that the maximum of S follows the maximum of atmospheric pressure. This fact coinsides exactly with our anticipation, because in general the volcanic activity reaches the maximum after the maximum of atmospheric pressure [2]. Next, since we have gotten numerical data about the variation of volcanic activities of Kujiu, we will study the rate of influence.

5. The rate of influence of atmospheric pressure

Comparing the coefficients of S_1 and S_2 term with that of P_1 and P_2 term, if we

assume that 25 tons spouted sulphur (a monthly amount) corresponds to 1mm Hg. variation of atmospheric pressure, the amount of S influenced by the total variation of atmospheric pressure will be expressed as follows;

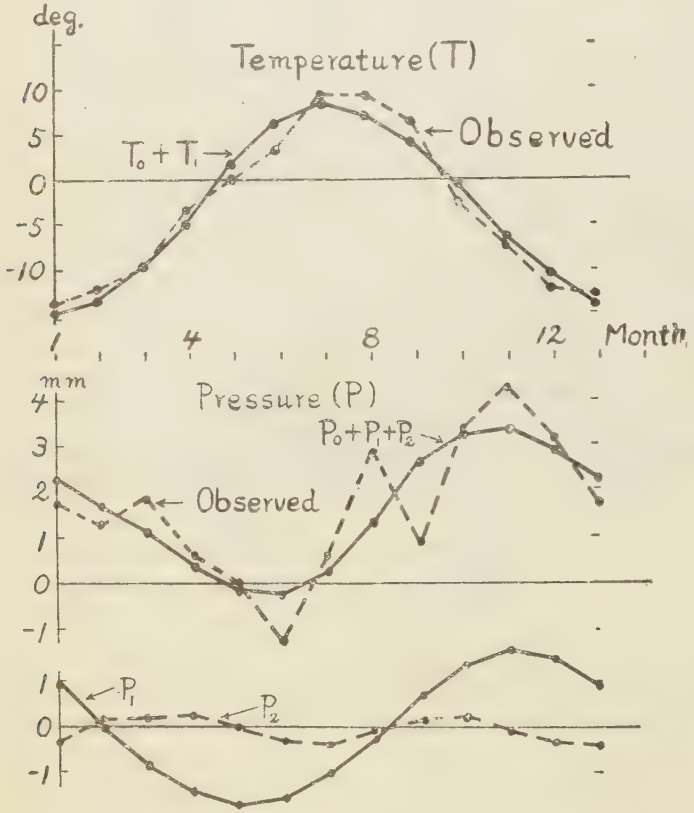

Fig. 2

$$45.0\cos(t-333)+8.4\cos(2t-280)$$

by subtraction from the observed amount, we have

$$-5.1\cos(t-333)+3.6\cos(2t-280)$$

or $5.1\cos(t-153)+3.6\cos(2t-280)$

Moreover, comparing the coefficients of above expression with W_1 and W_2 (the coefficients of expression of precipitation), we have $5.1/89.35 \fallingdotseq 3.6/69.35 \fallingdotseq 0.5/10$

Therefore the amount of S influenced by 10mm variation of preciptation is 0.5 ton and the phases of both variations nearly coinside.

Thus the annual variation of spouted sulphur or the annual volcanic variation of Kujiu depends chiefly on the variation of atmospheric pressure, and slightly on the precipitation, but both effects are in analogous manner. In this standpoint one may understand the diagram of S, and the infeuence of atomospheric pressure is 50 times that of precipitation. Therefore it is probable that in many cases the influence of precipitation is not distinct for covered by the influence of atmospheric pressure, and there exists a slight phase lag between the atmospheric pressure variation and its effect upon the spouted sulphur.

6. From the results just described, we may conclude as follows.

(1) The increase of S in March is due to the increse of P and W.

(2) The effect of decrease of atmospheric pressure in June is not distinct, because it is modified by the maximum annual variation of precipitation.

(3) In spite of the increase of atmospheric pressure in August, as the precipitation decreased markedly and the effect of precipitation had been great in July, no remarkable variation is recognised in August.

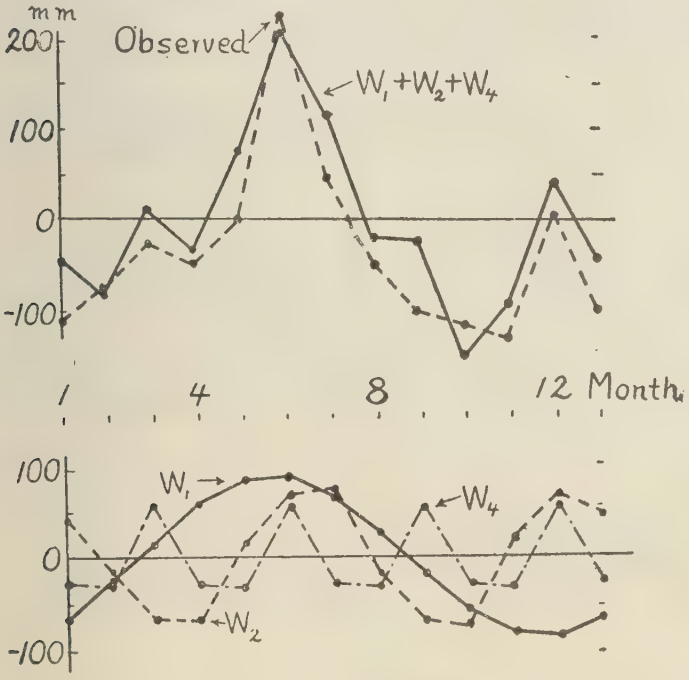

Fig. 3 Precipitation (W)

(4) The decrease of atmospheric pressure in September causes the concavaty of S curve.

(5) According to the writer's investigation, volcanic actions of Aso exist with the time of action grouping themselves in the trough of precipitation curve and in the trough of atmospheric temperature curve. It means that the volcanic actions occur in the creat of atmospheric pressure curve, and these facts are fairly in an agreement with the results of present studies.

(6) In the morning on May 23rd, 1948 "Taihei K1 Solfatara" at Kujiu began to act and effused about 100kg "Kuros" (black sulphur). At 10 O' clock on May 24th, (the following day), the first crater at Aso which had been calm spouting white smoke, erupted black smoke violently and trailed in the direction ENE. As shown in Fig. 4, it is evidently an example of phenomena which occur after the maximum variation of atmospheric pressure. The eruption of Aso occured one day later than that of Kujiu. This may be attributed reasonably to the fact that the depth of active source of the former is deeper than that of the latter. It is interesting that just when volcano had stored up internal energy enough, the variation of atmospheric pressure caused the eruption in the manner above described.

(7) At Kujiu sulphur mine, it is said that "In case of rain or typhoon the production of spouted sulphur decreases" [3]. This fact is easily understood as the influence of decrease of atmospheric pressure.

3. Conclusions

The main points of the present researches are summarised as follows;

1. The temperature variations have little or no influence on volcanic action.

2. Volcanic action depends chiefly on the variations of atmospheric pressure and partly on that of precipitation; the effect of the latter is about 1/50 that of the former.

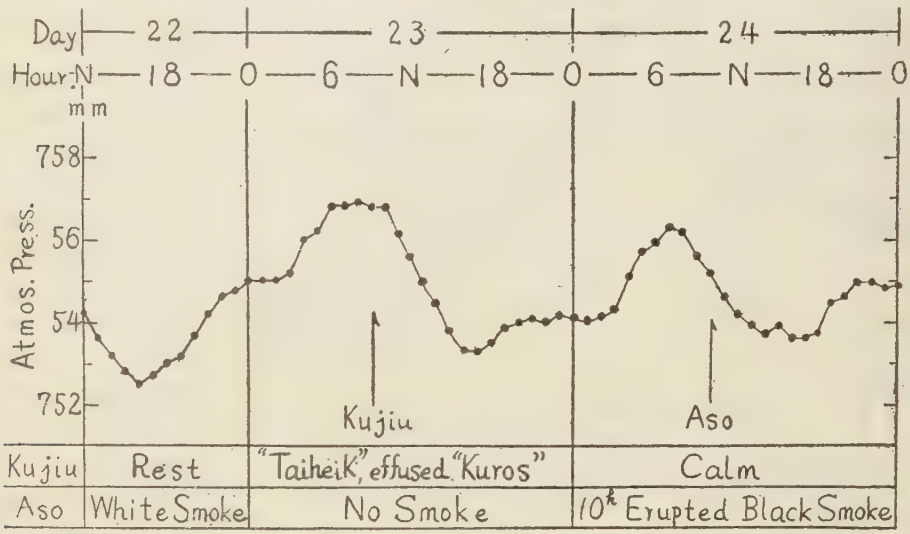

Fig. 1

3. The production of spouted sulphur is founded mostly on two elements viz. annual variation and semi-annual variation. The production increases in winter and decreases in summer.

4. In case of rain or typhoon the production of sulphur decreases owing to the decrease of atmospheric pressure; and in case of fine or travelling high atmospheric pressure it increases.

5. Arriving at the state where the stored energy has increased enough, this volcano begins to act violently under the influence of high atmospheric pressure and then erupts black sulphur.

6. 1mm variation of atmospheric pressure (monthly average) causes the variation of production of spouted sulphur at thr rate of 25 tons per month, and 10mm precipitation causes the variation 0.5 ton.

7. It will require a further research to learn how to make black sulphur erupt by varying the atmospheric pressure artificially. The writer's future programme is to study on diurnal and secular volcanic variations of Kujiu.

In concluding this paper the writers wish to express their hearty thanks to Mr. S. Kato for his endeavor in observation over a long period of time and also to B. Sc. Arita for his help in calculation and drawing.

A part of the cost of this research was defrayed from the Scientific Research Expenditure of the Department of Education. The writers wish to express their cordial thanks to the authorities concerned. This work was published as Physical Note No. 1 of Physical Institute Kumamoto University (June 1, 1950), and reported to the regualr meeting of Kyusyu Branch of Physical Society Japan (Dec. 1950).

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SOME STUDIES ON VOLCANO ASO AND KUJIU (PART 2)
ON THE PERIODICITY OF TWO ACTIVE VOLCANOES ASO AND KUJIU

Munetosi NAMBA Tosisato MUROTA and Tokanori MITSUI

(Received Januauy 31, 1952)

Abstract

In the present paper, the writers compared the results of analysis of the spouted sulphur of Volcano Kujiu with active periods of Volcano Aso.

1 Introduction

The sulphur mine on Volcano Kujiu with its active crater, is 1300m above the sea-level. It is said that the work of collecting the sulphur there has been carried on since about the Kyoho era (1716-)[1]. Draught—method has been employed from the 29th year of Meiji (1896) up to the present. Lately thorough researches have been accomplished of the volcano by Mitsui Investigation Association with Takanori Mitsui at its head.

Hereupon a complete table has been made of the sulphur output covering 50 years. The present writers, analysing this table from the geophysical standpoint, have ascertained 24, 12 and 6 years, respectively, as prominent in the active periods of the volcano, and further tried a comparison with them of the movements of the active crater of Naka-dake (Middle Hill) of Volcano Aso.

2. Variations in the Annual Rate of Sulphur Output

TABLE 1.

Year	Spouted Sulphur	Refined Sulphur	Year	Spouted Sulpnur	Refined Sulpnur	Year	Spouted Sulphur	Refined Sulphur	Year	Spouted Sulpnur	Refimed Sulphur
	ton	ton		ton	ton		ton	ton		ton	ton
1896	0	423	9	918	354	1921	90	688	34	560	557
97	201	345	10	1136	605	22	22	0	35	688	557
98	279	372	11	1255	998	23	18	388	36	302	520
99	315	182	12	1327	926	24	8	376	37	395	502
1900	343	453	13	1117	946	25	2	454	38	760	313
1	467	602	14	946	969	26	16	379	39	500	0
2	756	703	15	946	1028	27	99	468	40	81	278
3	1498	738	16	870	1236	28	360	316	41	343	308
4	2357	732	17	596	1072	29	306	506	42	349	354
5	1828	821	18	622	607	30	438	452	43	866	103
6	1817	830	19	846	597	31	305	502	44	594	204
7	1892	1057	1920	299	717	32	188	500	45	485	10
8	1363	514				33	275	589			

Table 1 shows the annual output of spouted sulphur, and refined sulphur from the deposits, on Volcano Kujiu. Refined sulphur is by nature remarkably subject to variation in output with the number of mobile hands. Spouted sulphur, on the other hand, is, it may safely be said, comparatively little affected by such a condition. It is from this view-point that the writers limit their investigations here to the spouted sulphur.

An examination in the curvatures of the yearly output of spouted sulphur clearly shows the rise and fall in the curvilinear course of growth.

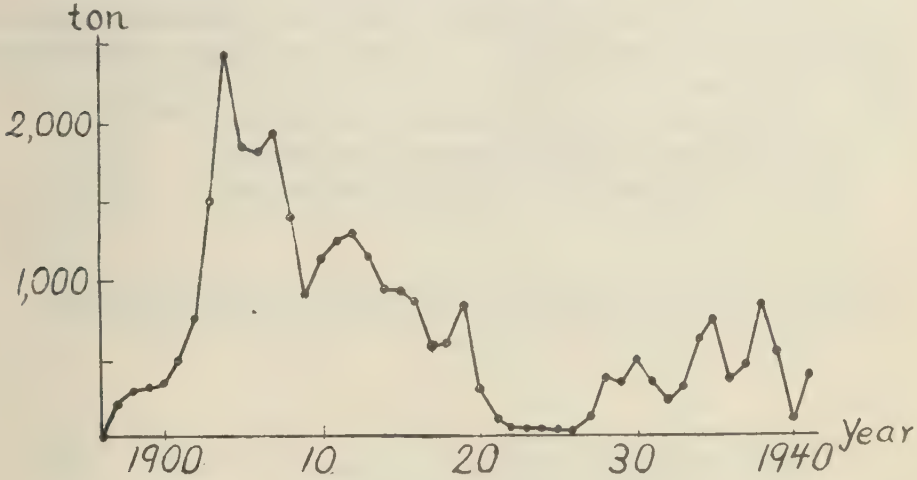

Fig. 1 Spouted Sulphur from Volcano Kujiu

When, this secular trend taken into consideration, the variations are calculated, the result is such as Fig. 2 indicates.

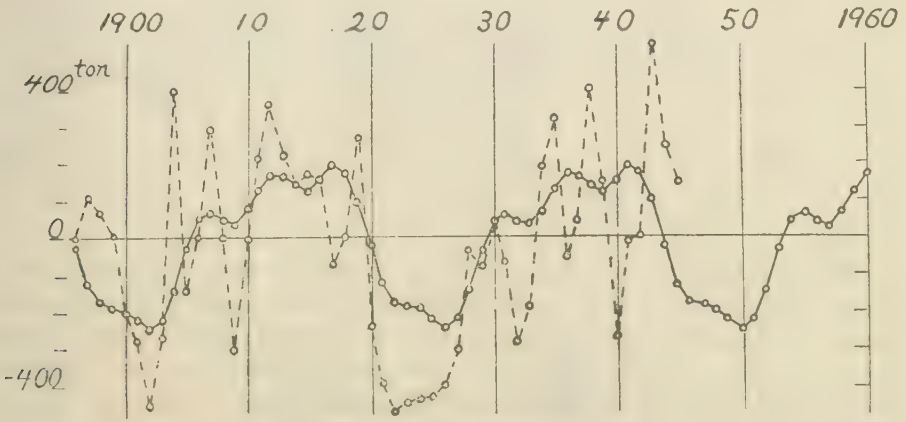

Fig. 2 Spouted Sulphur

Here we can distinctly discern periodic variations. Among them, extremely prominent are the following:

$$\begin{aligned} &\text{ton} \\ &195.3\ \cos\ (n2\pi/24-260) \\ &62.3\ \cos\ (n2\pi/12-301) \\ &49.7\ \cos\ (n2\pi/6\ -240) \end{aligned}$$

provided that n=0 is the year of 1896.

By this we see that the 24 years period has the broadest amplitude, which is almost 3 times as much as that of the 12 years one, or of the 6 years. Graphic details may be seen in Fig. 3. To summerise this:

(1) With the 24 years period, the variation is about 400 ton.
(2) With the 12 years period, the variation is about 120 ton.
(3) With the 6 years period, the variation is about 100 ton.

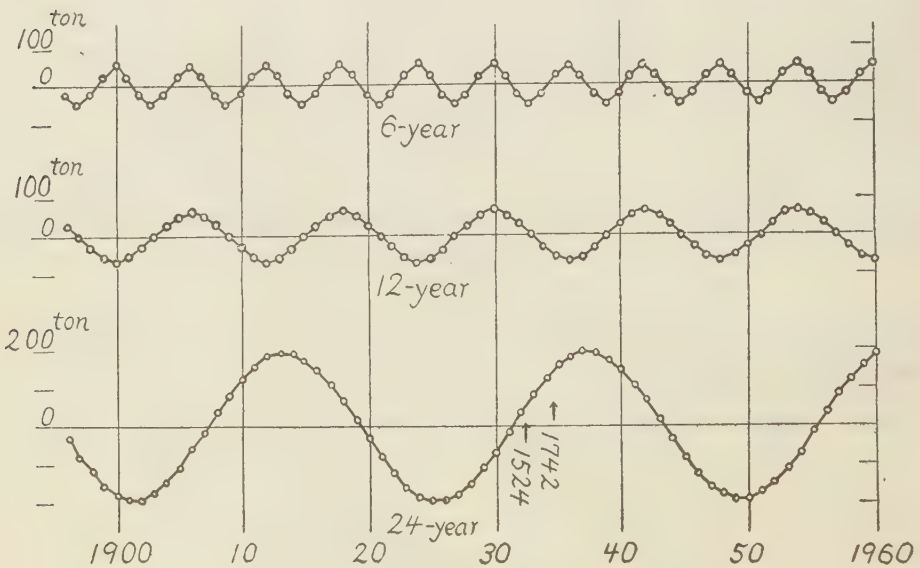

Fig. 3

These variations may be ascribed, in short, to the activities of the volcano. Whatever elements volcanic gases may be composed of, their erupted volume is in proportion to the activities of a volcano. As eruptions increase, so does spouted sulphur. Therefore it follows that the activities of Volcano Kujiu may be known by variations in yearly output of the spouted sulphur. That is, the periods above mentioned exactly signify the activities of Volcano Kujiu.

According to old records, the volcano of Kujiu was active both in 1524 and in 1743 [2]. The dates are indicated by arrows in the diagram. These correspond to the ascending part in the 24 years variation, and we are sure that the accounts on record almost coincide with the results of this analysis.

We now turn to variations in average temperature and rainfall of the year, and weigh them with those in activity. We find that

(1) Beyond the summit of variations in rainfall, rises the top of variations in activity.

(2) At the bottom of variations in temperature, rises the top of variations in activity.

3. Variations in Activity of Volcano Aso

One of the writers has already ascertained the principal periods of activity of Volcano Aso by analysing the variation in the rate of growth of the cedar growing there [3].

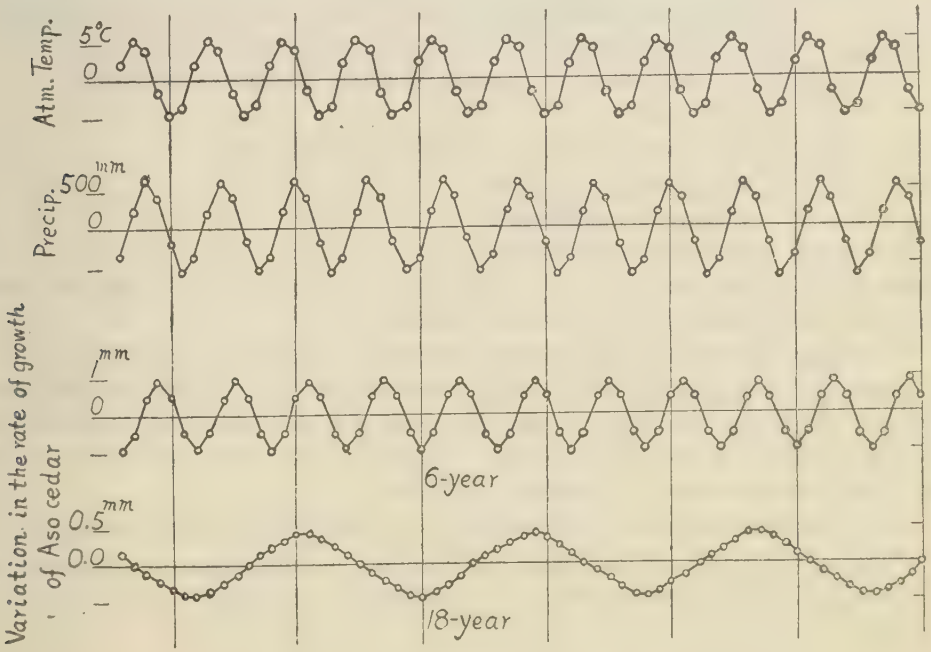

Fig. 4

Thereupon we know that

(1) Activities of the volcano begin at the end of the greatest of variations in rainfall, and stop in the middle of the ascending part; in other words, they are fierce at the bottom.

(2) As for variations in temperature, activities of the volcano are generally great at the bottom of variations.

(3) Judged from variations in the rate of growth of the Aso cedar, activities converge somewhere upon the height of variations.

As there are many data refer to, it is convenient to make good use of variations in the rate of growth of plants for the investigation into long period variations in volcanic activity. Surely variations in the rate of growth of the Aso cedar nearly coincide in curve with those in rainfall [4]. It is on this account that the writers compare and contrast it with volcanic activities of Aso.

4. Relations between two Volcanoes Aso and Kujiu

Judged from the foregoing results, Volcano Aso is quite at one with Volcano Kujiu in relation to variations both in rainfall and temperature. As rainfall increases with the rise in temperature, variations in temperature are naturally a little abvanced in pahse; and analytic examination results in the same thing.

Now, supposing the pressure of a volcano to be, P, its eruption wil lbe

$$dP/dt=aP.$$

Again, if outer pressure such as the variation in rainfall exists,

$$P=p-b\cos t. \qquad \text{If so,}$$

$$-dp/dt=ap+R\cos(t-w).$$

It is needless to remind one that $w=180°-h$.

Therefore the eruption must lag by w in phase [5]. In this matter, both volcanoes coincide with each other, the same analytic results produced.

Although both Aso and Kujiu are affected quite the same by outer pressure, the former is periodically active with the interval of 6, 11 and 18 years, and the latter with the interval of 6, 12 and 24 years. This signifies that the present sources of activity of these two volcanoes exist quitie independently of each other. Furthermore, the following points can be pointed out by the results produced by the present analysis.

(1) Both of the volcanoes have the periods of 11 and 12 years nearly the same in length with the sunspot cycle, and of 6 years half as much; and this give rise, it is feared, to a dangerous liability directly to connect the activities of these volcanoes with those of the sunspots.

(2) It is found out that both volcanoes are affected quite in the same way by variations in outer pressure such as those in rainfall. Accordingly it is surmised that they may start activities so closely one upon the other by some external condtiion of the same sort, when volcanic activities of both are about to reach the highest point, that is, when inner pressure is on the point of explosion.

(3) When the miximum of activity of one is side by side with the minimum of activity of the other, there may appear a phenomenon like a seesaw game. (Indicated in the diagram)

Under such a condition, we are apt to be impressed as if both of the volcanoes are directly connected underground with each other. But full examination in long period variations will tell us that it is a mere freak of phase differences.

5. Conclusion

Volcanic activities were very seldom observed quantitatively. And in this sense, research data concerning the activities of Volcano Kujiu may rightly be compared to a welcome rain in the burning sun. The writers have been enabled to ascertain the periodicity, the connection with Volcano Aso of this volcano, and even the characteristics of the volcano itself.

This work was carried out under the support of the Scientific Research Expenditure of the Department of Education, and published as Physical Note No. 2 of Physical Institute Kumamoto University (Nov. 30, 1950), and reported to the regular meeting of Kyusyu Branch of Physical Society Japan (Mar. 1951),

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SOME STUDIES ON VOLCANO ASO AND KUJIU (PART 3) ON THE SHAPE OF OLD ASO CRATER AND THE DISTRIBUTION OF THE CENTRAL CONES AND HOT SPRINGS

Munetosi NAMBA and Tosisato MUROTA

(Received January 31, 1952)

Abstract

The shape of the old crater of Aso being determined by harmonic analysis, and the result thereof applied to the distribution of the central cones and hot springs, it will be understood that the process of depression in Aso has continued on to the present state, making cracks which can be classified in two types.

1. Foreword

The old Aso crater, considered the greatest in the world, is certainly a depression caldera. This is what Mr. Johsei Iki positively states in No. 33 [1] of Earthquake Investigation Committee Report, the conclusiveness thereof Professor Ryuji Nomitsu [2] examined and affirmed geophysically by boring wells. No report, hower, as to the shape and extent has ever been submitted with any approach to certainty, opinions differing on these subjects. As regards the extent alone, the present co-writer, Namba, taking into consideration many reports so far given in, reckoned it nearly elliptical in shape, and found out the following values; about 25 km from south to north (major axis), about 19 km from east to west (minor axis), about 80 km in circumference, and about 370 square km in area [3]. In 1944, further, in cooperation with Mr. Tosiharu Etoh, and in 1945, with Yutaka Kiyosaki he determined the shape of the old crater by harmonic analysis. This time the writers try to examine minutely these results anew and find some new significance in the disposition of the central cones and hot springs scattered about in the old crater plains.

2. Harmonic Analysis of the Shape of the Old Crater

When, availing yourself of a topographical map on a one fifty thousandth scale issued by the Geographical Investigation Institute, you draw an east-west line from the cross-roads, in the ravine of Tateno, of Kumamoto-Miyazaki and Kumamoto-Ohita Roads, it will pass over another of Nangoh Road near the main shrine on the mountain summit, and reach the neighbourhood of the "route of Ohto" on the eastern crater walls. And again, when, with this turning point from Nangoh Road for an origin, and a Tateno line for a base, you draw radiant lines clockwise at every 15 degrees, and measure the distances of intersection of these with the crater cliffs, you will find 24 values. What you have measured by this method do

we term Type I. Next taking a line, which makes an angle W7.°5 N with the Tateno line, for a base, you have 24 values in the same way, that are termed Type II. Further, we propose to term Type III distances in the same radiant directions as in Type I, which consist in average values of three distances, namely, distances in Type I, and distances before and behind in Type II. These values analysed in the form

$$N=N_0+N_1\cos(x-\theta_1)+N_2\cos(2x-\theta_2)+N_3\cos(3x-\theta_3)+\cdots\cdots\cdots$$

and the smaller term N_4 and downwards ignored, they will be

Type I : $21.056+2.075\cos(x-44.6)+2.743\cos(2x-171.6)+1.499\cos(3x-241.4)$

Type II : $21.405+2.960\cos(x-50.6)+3.299\cos(2x-164.4)+1.530\cos(3x-233.8)$

Type III : $21.289+2.634\cos(x-54.5)+3.032\cos(2x-177.0)+1.431\cos(3x-251.2)$

Measured values, calculated values and probable errors are indicated in Table 1, 2 and 3:

TABLE 1. For Type I

Base Line	Observed Value	N_0+N_1	N_0+N_2	N_0+N_3	$N_0+N_1+N_2$	$N_0+N_1+N_3$	$N_0+N_1+N_2+N_3$
	cm						
0	19.40	22.534	18.342	20.339	19.820	21.817	19.103
1	19.90	22.860	18.906	19.618	20.710	21.422	19.272
2	21.60	23.064	20.046	19.740	22.054	21.748	20.738
3	22.90	23.131	21.457	20.632	23.532	22.707	23.108
4	23.62	23.056	22.760	21.773	24.760	23.773	25.474
5	28.85	22.845	23.607	22.494	25.396	24.283	26.834
6	25.96	22.512	23.770	22.372	25.226	23.828	26.542
7	22.58	22.080	23.206	21.480	24.230	22.504	24.654
8	24.60	21.578	22.066	20.339	22.588	20.861	21.871
9	20.10	21.041	20.655	19.618	20.640	19.603	19.202
10	14.94	20.504	19.352	19.740	18.800	19.188	17.484
11	16.81	20.005	18.505	20.632	17.454	19.581	17.030
12	18.60	19.578	18.342	21.773	16.864	20.295	17.531
13	19.32	19.252	18.906	22.494	17.102	20.690	18.540
14	18.45	19.048	20.046	22.372	18.038	20.364	19.354
15	19.30	18.981	21.457	21.480	19.382	19.405	19.806
16	21.40	19.056	22.760	20.339	20.760	18.339	20.043
17	20.24	19.267	23.607	19.618	21.818	17.829	20.380
18	19.45	19.600	23.770	19.740	22.314	18.284	20.998
19	21.80	20.032	23.206	20.632	22.182	19.608	21.758
20	24.00	20.534	22.066	21.773	21.544	21.251	22.261
21	21.75	21.071	20.655	22.494	20.670	22.509	22.108
22	21.47	21.608	19.352	22.372	19.904	22.924	21.220
23	18.30	22.107	18.505	21.480	19.556	22.531	19.980
Probable Error (%)		7.7	6.0	9.1	5.4	7.6	4.0

TABLE 2. For Type II

Base Line	Observed Value	N_0+N_1	N_0+N_2	N_0+N_3	$N_0+N_1+N_2$	$N_0+N_1+N_3$	$N_0+N_1+N_2+N_3$
	cm						
0	19.00	23.285	18.228	20.501	20.108	22.381	19.204
1	21.72	23.813	19.039	19.893	21.507	22.301	19.995
2	22.87	24.176	20.587	20.170	23.358	22.941	22.123
3	23.14	24.352	22.295	21.171	25.242	24.118	25.008
4	28.34	24.325	23.763	22.309	26.683	25.229	27.587
5	28.80	24.101	24.601	22.917	27.237	25.613	28.809
6	26.00	23.692	24.582	22.640	26.869	24.927	28.104
7	27.70	23.127	23.711	21.639	25.433	23.361	25.667
8	24.35	22.445	22.232	20.501	23.263	21.541	22.359

9	17.30	21.692	20.515	19.893	20.802	20.180	19.290
10	16.00	20.920	19.047	20.170	18.562	19.685	17.327
11	16.85	20.181	18.209	21.171	16.985	19.947	16.751
12	19.57	19.525	18.228	22.309	16.348	20.429	17.252
13	17.40	18.997	19.069	22.917	16.691	20.509	18.203
14	17.72	18.634	20.587	22.640	17.816	19.869	19.051
15	20.70	18.458	22.295	21.639	19.348	18.692	19.582
16	20.45	18.485	23.763	20.501	20.843	17.581	19.939
17	19.76	18.709	24.601	19.893	21.905	17.197	20.393
18	20.05	19.118	24.582	20.170	22.295	17.883	21.060
19	21.35	19.683	23.711	21.171	21.989	19.449	21.755
20	23.40	20.365	22.223	22.309	21.183	21.269	22.087
21	22.66	21.118	20.515	22.817	20.228	22.630	21.740
22	19.10	21.890	19.047	22.640	19.532	23.125	20.767
23	18.90	22.629	18.209	21.639	19.433	22.863	19.667
Probable Error (%)		10.1	9.3	11.0	5.5	9.5	4.5

TABLE 3. For Type III

Base Line	Observed Value	N_0+N_1	N_0+N_2	N_0+N_3	$N_0+N_1+N_2$	$N_0+N_1+N_3$	$N_0+N_1+N_2+N_3$
	cm						
0	19.10	22.828	18.232	20.814	19.785	22.387	19.324
1	20.21	23.334	18.703	20.011	20.762	22.070	19.498
2	22.06	23.700	19.862	19.948	22.287	22.373	20.960
3	22.97	23.900	21.400	20.663	24.025	23.288	23.413
4	25.03	23.921	22.904	21.736	25.550	24.382	26.011
5	28.66	23.763	23.973	22.539	26.461	25.027	27.725
6	26.92	23.434	24.318	22.602	26.477	34.761	27.804
7	25.43	22.959	23.847	21.887	25.531	23.571	26.143
8	25.55	22.369	22.688	20.814	23.732	21.908	23.321
9	20.58	21.704	21.159	20.011	21.579	20.440	20.315
10	16.08	21.010	19.646	19.948	19.381	19.683	18.054
11	16.55	20.334	18.577	20.663	17.636	19.722	17.024
12	18.34	19.722	18.232	21.736	16.679	20.183	17.140
13	18.76	19.216	18.703	22.539	16.644	20.480	17.908
14	17.52	18.850	19.862	22.602	17.437	20.177	18.764
15	19.24	18.650	21.400	21.887	18.775	19.262	19.387
16	20.85	18.629	22.904	20.814	20.258	18.168	19.797
17	20.15	18.787	23.973	20.011	21.485	17.523	20.221
18	19.75	19.116	24.318	19.948	22.159	17.789	20.832
19	21.27	19.591	23.847	20.663	22.163	18.979	21.551
20	23.12	20.181	22.688	21.736	21.594	20.642	22.055
21	22.60	20.846	21.159	22.539	20.721	22.110	21.985
22	21.08	21.540	19.646	22.602	19.911	22.867	21.238
23	18.77	22.216	18.577	21.887	19.518	22.828	20.130
Probable Error (%)		8.7	7.8	9.7	4.6	8.1	3.3

According to these tables,

$21.056+2.075\ cos\ (x-44.6)\pm 7.7\%$ (Type I, 1/50,000)

is best applicable, when the old Aso crater is considered as almost circular in shape (N_0+N_1). The centre of this circle is in the position of 1.038 km NW from the origin. It has a diameter of 21.056 km importance attached on the greatest term N_2, and approximate ellipse (N_0+N_2) supposed

$21.056+2.743\ cos\ (2x-171.6)\ \pm 6.0\%$ (Type I, 1/50,000)

is best, the major axis (from south to north) being 23.770 km, and the minor axis (from east to west) 18.342 km. Both in case of taking the terms up to the second ($N_0+N_1+N_2$), and those up to the third ($N_0+N_1+N_2+N_3$), Type III contains the least of probable errors.

$21.289+2.634\ cos\ (x-54.5)+3.032\ cos\ (2x-177.0)\pm 4.6\%$

$21.289+2.634\ cos\ (x-54.5)+3.032\ cos\ (2x-177.0)+1.431\ cos\ (3x-251.2)\pm 3.3\%$

(Type III, 1/50,000)

In any of the Types, the second term N_2 is the greatest. The old crater has hitherto been regarded approximately elliptical in shape. The result of harmonic analysis nearly coincides with this opinion. Next, in each several shapes, the position of the centre, axes, circumference and area may be indicated approximately as in the following table.

TABLE 4.

Shape	Type	Center	Diameter	Circumference	Area
Circle (N_0+N_1)	I	1.04 km NW from the origin	km 2R=21.06	km 66.15	km^2 348.2
Ellipse (N_0+N_2)	I	the origin	2a=23.77 2b=18.34	66.56	342.4
Distorted Ellipse ($N_0+N_1+N_2$)	I	0.89 km N 15° W from the origin		67.10	347.1
	III	1.06 km N from the origin	2a=24.32 2b=18.23	68.20	356.2
Distorted Oval ($N_0+N_1+N_2+N_3$)	I	1.61 km N 15° W from the origin		67.60	348.9
	III	1.74 km N from the origin		68.90	357.1

Take the terms up to the third term ($N_0+N_1+N_2+N_3$) in Type I, and 4% of probable errors may be anticipated. In this degree of accuracy, the shape of the old crater is indicated as a distorted oval-shape, with its major axis leaning a little to west, and, measured in the central part, about 24 km from south to north, about 18 km from east to west, about 70 km in circumference, and 350 square km in area. It is bigger than about half of Lake Biwa, and about two times bigger than Kasumi-ga-Ura.

3. The Distribution of the Central Cones and Hot Springs

Of the western half of the central cones and hot springs in Aso, the distribution was pointed out by Professor Ryuji Nomitsu, [4] in 1939, to be on the concentric circles and on the radiant lines of the depression caldera. The present writers, taking into consideration the restlts of harmonic analysis, examine anew the distribution of the central cones and hot springs.

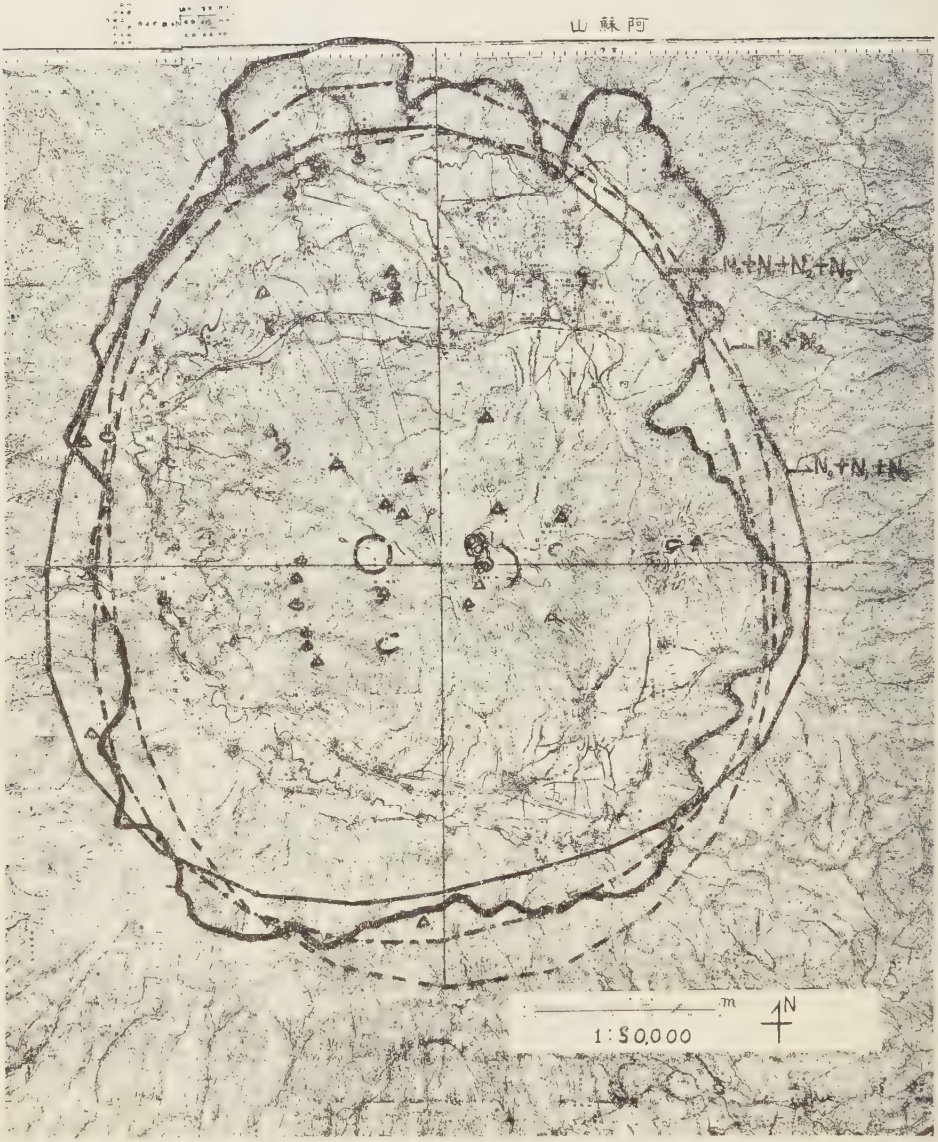

Fig. 1 Map of Volcano Aso

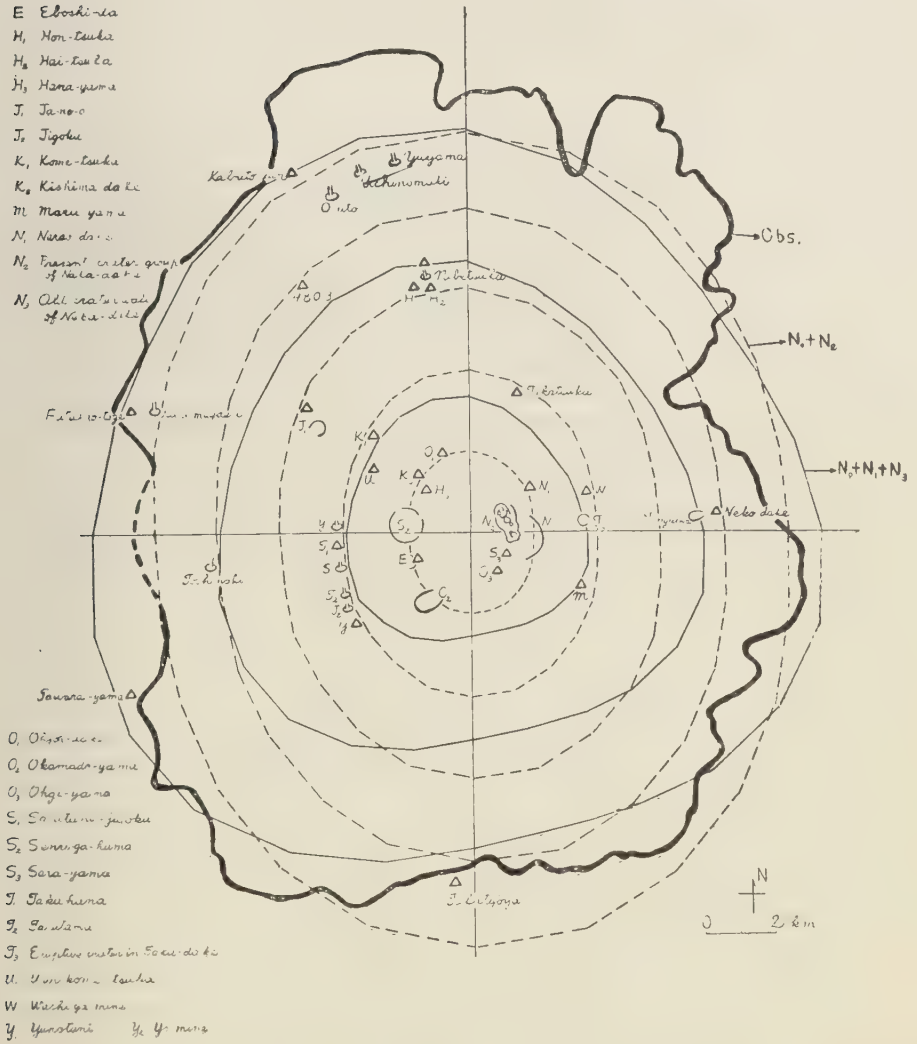

Fig. 2 Distribution of the central cones and hot spings

As to the forms N_0+N_2 and $N_0+N_1+N_3$, as is seen from the table of probable errors, the best results can be expected of Tape I. Therefore the writers take up this Type and discuss it. (Cf. Fig. 1 and 2) Draw over the crater of the shape $N_0+N_1+N_3$ (distorted diamond-shape) concentric figures, of which each line passing the points of trisection of the lines rediating from the centre, and

(1) the first line falls on the crater bank, on which there are to be found the epicenter of Tateno ravine, and remarkable landslips from Futaeno-toge to Kabuto-iwa;

(2) on the second line, Tochinoki hot springs, Nibetsuka, and Tengu-iwa in Nekodake;

(3) on the third line, Uwakome-tsuka, the eruptive crater in Takadake, Maruyama, Takahana, Yunotani hot springs, Jigoku hot springs, Tarutama hot springs, Sarudani-Jigoku. Again, draw over the crater of the shape N_0+N_2 (ellipse) concentric figures, of which each line passing the points which divide the radiant lines into five equal parts, and you will find situated

(1) on the first line, Kurumagaeri hot springs, spontaneous hot springs of Orito, Uchinomaki hot springs, Yuyama spontaneous hotsprings;

(2) on the second line, Tochinoki hot springs, 480.3 hill, Nekodake;

(3) on the third line, Ja-no-o volcano, Hontsuka and Haitsuka;

(4) on the fourth line, volcanoes of Kometsuka, Takatsuka, Washiga-mine, Yonine and Takahana; Jigoku hot springs, Tarutama hot springs, Yunotani hot springs, and Sarutani-jigoku;

(5) on the fifth line, the eruptive crater in Senriga-hama, Hanayama, Kishimadake, Ohojoh-dake, Narao-dake Okamado-yama, and Eboshi-dake;

(6) on the outer line of half the innermost figure, and nearly parallel to the old crater wall of Nakadake, the present crater group ear-like in shape of Nakadake; and, besides, Ohgiyama and Sarayama.

The above examination apparently is not so wellgrounded, yet the results are too lucky for the writers. They then make a trial at theorization, and realize the facts that, when the greater Aso was depressed

(1) its depression and cracking in the earlier period was in the form of $N_0+N_1+N_3$, and had concentric figures with the form; some eruption might take place on their lines;

(2) and in the next period, they took place in the form of N_0+N_2 and concentrically with it, and on their lines arose probably some of the cones and hot springs.

Thus considered, the cones and hot springs along the cracks in the form of $N_0+N_1+N_3$ may equally belong to the comparatively old period, and those on the N_0+N_2 cracks equally to the comparatively new. And all of them can be gathared altogether on the lines of both of these above figures. It is then too much to suppose that a process of depression such as the above took place? Furthermore, the existence of the present ear-shaped crater group of Nakadake, nearly parallel to the old crater wall of Nakadake, is suggestive, in all probabi-

lity, of the course and transition of the volcanic activities. When the cones and hot springs in the old crater, promiscously distributed as they seem, are arranged in such a method as above-mentioned, they prove to have made a certain regular development. This may safely be asserted.

4. Conclusion

On examining the results of harmonic analysis, turning attention chiefly to the mere shape of the old Aso crater, the writers find it nearly a distorted ovalness. Ellipse is prominent as a regular shape, and roundness next. In any case, it is about 70 km in circumference, and about 350 square km in area.

Among the cones and hot springs, those belonging to the comparatively earlier age seem to be situated on the cracks in the $N_0+N_1+N_3$ type, and those to the comparatively later, on the cracks in the N_0+N_2 type. The writers are very much contented if they may safely assert that the process of depression proceed, making cracks which can be classified in two types as above-mentioned, until it has reached the present state. The above results will be checked quite agreeably in the following paper "On Volcano Kilauea".

This work was reported to the regular meeting of Kyusyu Branch of Physical Society Japan (Sep. 1951).

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ON THE ELECTRICAL CHARACTERISTICS OF CONCRETE AND OTHERS (I)

Shigeichi FUJITA

(Received January 31, 1952)

Abstract

We have measured the electric resistances of concrete and others by a special method using direct current and ascertained their particular electric properties. Especially it is found that the resistances of some concrete, cemented stones and cemented metals are very different when the direction of electric current is reversed. We report here the method of measurement employed and some results of the experiments.

1. Method of Measurement

Among the several methods [1], [2], [3], [4] of measuring high resistances, we have employed the next method which is considered a modified direct deflection method. It is as follows.

In the first place, a battery (1 or 2 dry cells) E, a known resistance Q (about $10^4\Omega$), an unknown high resistance X to be measured and a galvanometer (resistance G) are connected in series through a key K as shown in Fig. 1. Making the circuit, the reading of the deflection of the galvanometer is taken. If the E. M. F. of the battery is E, the strength of the electric current through the galvanometer is given by

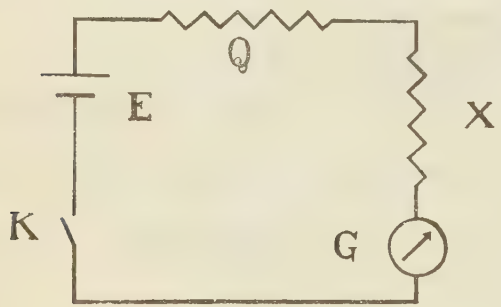

Fig. 1.

$$i = \frac{E}{Q+X+G} \qquad (1)$$

In the second place, the unknown resistance X is replaced by a variable known resistance P and a small fixed resistance R ($100\Omega \cdots\cdots 1\Omega$) is connected in parallel to it and the galvanometer as shown in Fig. 2. P should be adjusted so as to show the same deflection of the galvanometer as in the first case. Then the strength of the electric current through the galvanometer is

$$i' = \frac{ER}{(P+G)\,R+Q\,(P+G+R)} = i \qquad (2)$$

From (1) and (2) we get

$$X = P + \frac{Q}{R}\,(P+G) \qquad (3)$$

When X is smaller than $10^8\,\Omega$, P may be smaller than $10^4\,\Omega$, Q about $10^4\,\Omega$ and R fixed at 100Ω, 10Ω or 1Ω. When X is greater than $10^8\,\Omega$, it is necessary to use resistances greater than $10^4\,\Omega$ for Q and P. In the circuit given in Fig. 2, if the value of P is changed successively, the deflection of galvanometer changes also with the value of P, and we can plot a curve expressing the relation between P and the deflection of galvanometer. As we can calculate the value of X by (3) when the value of P is given, a curve is plotted which expresses the relation between X and the deflection of galvanometer. By using this curve we can easily obtain the value of X only by reading the deflection of galvanometer in the first experiment. Thus we can measure the values of many unknown high resistances in a short time.

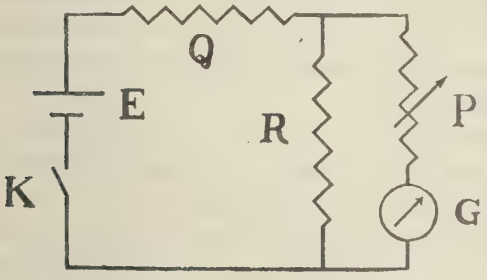

Fig. 2.

From (2), the variation of i' which is caused by the variation of P is given in the form

$$\frac{\delta i'}{i'} = -\frac{\delta P}{P+G+\frac{RQ}{R+Q}} \tag{4}$$

In this expression, as G and RQ/R+Q are very small compared to P, we know that the value of P and therefore the value of X are accurately determined.

2. Removal of Contact Resistance

To measure a high resistance by direct current, it is usual to put the specimen between two metallic electrodes. But it is difficult to bring the electrodes into sufficiently close contact with the end surfaces of specimen, and the contact resistance always affects the experimental results. [5], [6], [7]

To remove this error, several methods are tried. For example, tin foil, mercury or filter paper wetted by salt water is inserted between metal plates and the specimen. But we experienced that tin foil or mercury does not make close electric contact, and salt water produces electromotive force in complex manner. We have, therefore, kneaded graphite powder by very dilute glue-water in paste and have plastered it at the end surfaces of the specimen, dried it up sufficiently. By this procedure, the effect of contact resistance has been removed.

3. Examples of the Mesaurement

Using the above mehod we have measured the resistances of concrete, stone, wood, brick and some of the volcanic rocks. Almost all of them have the complex properties of

resistance for the strength of current, the time of flow of current, the applied E. M. F., humidity and temperature. To simplify the problem, we have always used 2.6 volts as the E. M. F. of the battery, and have taken the readings of galvanometer at one minute after the E. M. F. is applied. It is observed that the effect of humidity is greater than the effect of temperature, but these effects are complex and it seems that the moisture contained in the specimen has great effect on the resistance.

Besides of it, some of the specimen have different resistances when the direction of the electric current is reversed. The differences are very great in some cases. Conventionally we affix marks α and β to the two ends of the specimen and indicate by $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ that the current flows from α to β and from β to α respectively. The resistances of some concrete columns are definitely different in these both cases as shown in Table II. In Table I the qualities of specimens used are given.

The resitances given in Table II are those measured just at one minute after E. M. F. was applied. We see the changes of values by the changes of humidity and temperature.

Table I. Table of specimens measured.

No.	kind and form	composition	
	concrete circular column	cement: sand: gravel	water/cement
(1)	(length 10cm, diameter 5cm)	1 : 2 : 2.5	0.60
(2)	〃	〃	〃
(3)	〃	〃	〃
(4)	〃	〃	〃
(5)	〃	〃	〃
(6)	〃	〃	〃
	concrete square column	cement: sand	water/cement
(7)	(length 16cm, width 4cm)	1 : 2	0.50
(8)	〃	〃	〃
(9)	〃	〃	〃
(11)	〃	1 : 1	0.45
(12)	〃	1 : 3	0.60
(13)	〃	1 : 4	0.80
(14)	〃	1 ; 5	0.80
	concrete circular column	cement: sand: gravel	water/cement
(15)	(length 20cm, diameter 5cm)	1 : 2 : 2.5	0.60
(16)	〃	〃	〃

(25)	Square column of stone	(Shimasaki-ishi)	length 9.0cm,	width 3.2cm
(26)	〃	〃	〃 5.9cm,	〃 3.2cm
(27)	〃	(Hai-ishi)	〃 9.2cm,	〃 3.2cm
(28)	〃	〃	〃 6.0cm,	〃 3.1cm
(29)	〃	(Midorikawa-ishi)	〃 5.0cm,	〃 3.0cm
(30)	Column of cemented stone	(Midorikawa-ishi)	〃 3.0cm,	〃 3.0cm
(31)	〃	〃	〃 3.0cm,	〃 4.2cm

Table II_1. Resistance of Concrete.

Specimen	Resistance ($10^4\ \Omega$)					
	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$
(1)	4.39	11.7	4.79	11.2	5.01	11.4
(2)	6.74	3.05	7.22	3.48	7.09	3.45
(3)	7.82	26.5	7.82	24.0	9.28	22.3
(4)	10.61	13.5	11.7	14.1	14.4	14.1
(5)	3.01	4.79	3.10	4.98	3.39	5.82
(6)	3.18	3.56	3.25	3.68	3.78	3.99
(7)	9.10	7.82	9.58	8.26	11.4	9.10
(8)	8.61	14.7	8.99	15.0	11.0	16.6
(9)	10.24	16.6	11.20	16.25	13.8	16.6
Temperarure	26°.5C		26°.8C		27°.1C	
Humidity	82%		80%		78%	
Vapour tension	20.8 mmHg		20.7 mmHg		20.8 mmHg	

Specimen	Resistances ($10^4\ \Omega$)					
	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$
(1)	5.22	13.0	5.55	11.7	5.18	11.2
(2)	8.18	3.97	7.82	4.59	7.35	4.01
(3)	12.4	22.3	11.2	20.5	9.90	18.3
(4)	14.1	18.8	13.0	16.6	12.2	15.0
(5)	3.20	5.50	3.07	5.13	3.07	4.59
(6)	4.30	5.69	4.18	5.01	4.12	4.35
(7)	17.9	13.0	16.6	12.7	15.7	11.7
(8)	17.0	24.0	16.3	21.5	15.7	20.0
(9)	20.5	23.1	18.9	21.5	19.4	20.0
Temperature	29°.0C		29°.5C		29°.6C	
Humidity	78%		77%		77%	
Vapour tension	22.8 mmHg		23.6 mmHg		23.8 mmHg	

Table II_2. Resistance of Concrete (11)……(16) and Stones (25)……(28).

Specimen	Resistance ($10^4\ \Omega$)					
	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$
(11)	8.25	8.38	6.98	7.08	8.40	9.20
(12)	20.0	20.8	19.7	20.2	25.6	25.6
(13)	44.4	43.4	41.0	41.2	48.8	50.2
(14)	81.4	80.4	75.8	75.2	89.4	95.6
(15)	10.3	13.4	9.4	12.5	11.3	16.4
(16)	19.9	11.7	18.0	11.1	21.4	13.9
(25)	33.8	33.0	32.9	32.6	35.6	36.4
(26)	27.8	28.5	27.5	28.3	29.4	31.8
(27)	527	505	515	496	564	545
(28)	421	404	412	391	450	442
Temperature	23°.0C		23°.6C		21°.2C	
Humidity	80%		83%		82%	
Vapour tension	16.8 mmHg		18.0 mmHg		15.2 mmHg	

Specimen	Resistances ($10^4\ \Omega$)					
	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$
(11)	8.25	9.40	11.6	13.4	11.3	11.5
(12)	25.1	26.0	41.1	41.6	38.5	38.5
(13)	49.8	54.8	83.6	90.8	83.0	93.0
(14)	91.6	98.0	151.0	151.0	140.0	141.0
(15)	11.4	16.2	16.5	23.8	16.3	21.7
(16)	20.2	14.0	33.8	21.7	30.0	20.2
(25)	36.6	38.2	53.0	53.4	51.9	52.2
(26)	38.0	33.8	44.7	47.9	44.7	46.7
(27)	615	606	871	826	800	780
(28)	486	472	643	650	636	636
Temperature	21°.5C		18°.5C		18°.6C	
Humidity	83%		78%		76%	
Vapour tension	15.6 mmHg		12.3 mmHg		12.1 mmHg	

Concrete columns (1), (2) and (3) have the great differences of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$). (4), (5) and (6) have the smaller differences in spite of the same proportion of mixture. Though there must be various causes as for the differences of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$), the next case should be considered as an effective one. The concrete columns were made in erect position, so that the upper and lower parts of cement just contact to the pebbles contained in concrete brock would have a little difference in composition, and the contact potential differences between cement and pebbles would differ at upper and lower surfaces of pebbles. As there are many pebbles in concrete, the differences would be summed up to a large value, causing the differences of resistances in two directions ($\alpha\rightarrow\beta, \beta\rightarrow\alpha$).

In concrete, containing no pebbles, the grains of sand would have similar action as the pebbles. But as the grains of sand are very small compared to the pebbles, the differences in compositions of cement at upper and lower parts of the grains of sand would be also very small. The difference of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$) in this case is, therefore, smaller than that in concrete containing gravels. In fact, concrete columns (7), (8) and (9) which contain no gravels have smaller differences than (1), (2) and (3) as seen in the Table II_1.

As the proportion of sand in concrete increases, the resistance increases as seen in (11), (12), (13) and (14) in the Table II_2. The concretes (15) and (16) which contain gravels show difference of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$).

The columns of natural stones have no difference as seen by (25)···(28). In the cases of natural stones we also see large effect of humidity and temperature, which suggests that comparatively large portion of the electric current flows along the surfaces.

To ascertain the idea that the difference of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$) occurs where cement contacts with stone, we have measured the resistances of stones (30) and (31) which have a layer of cement between one end of stones and one electrode, and of stone (29) which has no cement layer. For comparison we have also measured the resistances

of specimens (1), (2), (3), and (4) simultaneously. The result are shown in Table II$_3$.

Table II$_3$. Resistances of Concrete (1)···(4) and Stones (29), (30), (31).

Specimen	Resistances ($10^4\,\Omega$)			
	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$	$\alpha\rightarrow\beta$	$\beta\rightarrow\alpha$
(1)	7.4	17.1	7.4	17.1
(2)	14.3	6.56	14.2	6.7
(3)	16.6	45.3	16.3	43.4
(4)	12.9	15.9	12.6	16.0
(29)	80.4 (1)	79.0 (2)	80.0 (1)	82.0 (2)
(30)	73.6 (1)	85.1 (2)	70.0 (1)	88.2 (2)
(31)	20.7	25.1	20.3	26.1
Temperature	13°.3C		13°.3C	
Humidity	86%		84%	
Vapour tension	9.8 mmHg		9.5 mmHg	

(1) Current from cement layer to stone
(2) Current from stone to cement layer

As is expected, (1), (2), (3), and (4) show large differences of resistances in two directions ($\alpha\rightarrow\beta$, $\beta\rightarrow\alpha$), while (29) shows no difference and (30), (31) show differences about 20~30%, which are comparatively samll, but it is considered to be reasonable as the contact of cement and stone occurs only in one surface.

From these facts it is concluded that the cause of the difference of electric resistance of concrete in one direction and the reverse direction is at the contact surfaces of cement and stones.

As an example to show the fact that the moisture in the specimen has a important effect in the electric resistance, the resistances of wood are given in Table III which are measured day by day after they are cut off from a living tree. The result shows rapid increase with time elapsed. (1'), (2'), (3'), (4'), (5'), and (6') are the circular columns of wood

Table III. Resistances of Wood.
The measurement was begun at the next day after the wood columns were cut off.

Date	Sep. 7, 1950	Sep. 8, 1950	Sep. 9, 1950	Sep. 10, 1950
Temperature	29°.4C	29°.0C	28°.0C	27°.8C
Humidity	77%	77%	74%	80%
Vapour tension	23.3 mmHg	22.9 mmHg	20.8 mmHg	22.3 mmHg
Specimen	Resistances ($10^4\,\Omega$)			
(1')	0.640	0.858	1.21	2.22
(2')	0.737	0.924	1.28	2.06
(3')	0.605	0.804	1.16	1.77
(4')	0.731	0.929	1.46	2.36
(5')	1.60	2.77	4.56	6.20
(6')	1.35	2.39	4.10	6.14

which were cut off from a branch of a bead tree.

Their dimensions are:

	(1')	(2')	(3')	(4')	(5')	(6')
diameter	5.2cm	5.0	5.4	5.0	5.0	5.1
length	7.5cm	7.9	6.1	6.2	7.5	6.4

(1'), (2'), (3') and (4') have bark,

(5') and (6') have no bark.

The wood without bark has a large resistance and shows rapid increase of resistance compared to that with bark, so it seems that the large portion of electric current flows through the bark, and the evaporation of water is prevented by it.

4. Conclusion

By a simple, special method of measurement of high resistances, the resistances of concrete, stone and wood were measured and some knowledges about their complex electrical properties and their characteristics were acquired. Expecially it was showed that the resistances of some concrete columns in one direction are definitely different from the resistances in the reverse direction while stone and wood have no such difference, and that it is difficult to maintain the same properties, as they accept strong influences from humidity, temperature and others. Lately we have obtained some more definite knowledges about their electrical characteristics, so that we will report the details in the next opportunity.

In conclusion, I wish to express sincere gratitude to Prof. Nakamura and to Prof. Namba for their kind guidance and encouragement, to Prof. Harada for his kindness of making concrete columns, to Mr. Kaminishi for his co-operation in measurement and to other colleagues in the laboratory for their discussion and encouragement.

This report was read Dec. 1950 at the regular meeting of Kyushu Branch of the Physical Society of Japan.

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MAGNETOSTRICTION OF PERMINVAR-ALLOY

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1. Introduction

We reported the results of measurements of the longitudinal magnetostriction in nickel and iron. [1] As the perminvar-alloy has a large field range of the initial permeability, we take measurement of the longitudinal magnetostriction of this alloy in the initial permeability range.

At first, the magnetostriction in a alternate magnetic field is described.

2. In the alternate magnetic field.

(i) Principle and method of measurement.

Principle and method of measurement are the same as described in the previous paper. The principal parts of the apparatus are shown shematically in Fig. I.

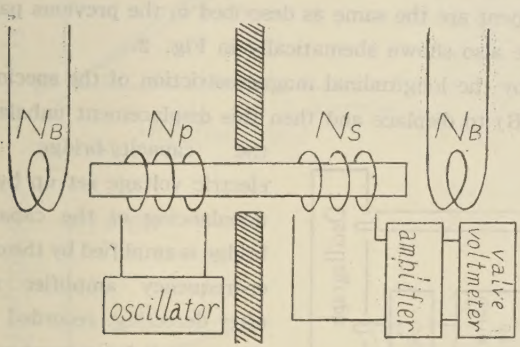

Fig. 1

The left hand side of the specimen is magnetized by the alternating-current which is supplied by the beat-oscillator but, on the other hand, the right hand side of the specimen is left in unmagnetized as the result of shielding. When the frequency of the alternating-current approaches the frequency of the mechanical vibration of the specimen, the right hand side of the specimen begins to vibrate longitudinally and vibrates most intensely at the same frequency as that of the mechanical vibration of the specimen, due to the mechanical resonance. As the result of the Villari-effect, the electric voltage is induced in the coil Ns, which is proportional to the longitudinal magnetostriction as shown from the thermodynamics. The induced voltage is amplified by the amplifier and then measured by the valve-voltmeter.

(ii) Specimen.

The specimen is a round bar of Fe-Ni-Co alloy, 5 mm in diameter and 46 cm in length, containing about 45% Ni and 24% Co.

(iii) Result.

As in the case of nickel, the longitudinal magnetostriction is observed even in the initial permeability range and the magnitude of the magnetostriction is greatly affected by heat-treatment and method of demagnetization. As the magnetostriction decreases to zero by demagnetizing the specimen by a alternate magnetic field of a large amplitude, the measurement is taken in a state in which the specimen is demagnetized by heating.

The initial permeability is measured ballistically by the method of reversals.

For an example, when the specimen is annealed at about 950°C for 2 hours and then cooled to room temperatue with the furnace, the induced voltage is observed by the valve-voltmeter. The magnitude of the induced voltage is greatly affected by annealing temperature and time to which the specimen is subjected, and this is considered to be caused chiefly by internal stress.

The heat-treatment of the so-called "baking" decreases the magnitude of the induced voltage in the initial permeability range. When the specimen above mentioned is annealed furthermore at 460 °C for about 80 hours, the magnitude of the induced voltage decreases by about 80% and the initial permeability decreases also by about 30%. The difference in these two cases is considered to be caused chiefly by the condition of the superstructure.

3. In the static magnetic field.

(i) Principle and method of measurement.

Principle and method of measurement are the same as described in the previous paper. The principal parts of the apparatus are also shown shematically in Fig. 2.

The change in length, produced by the longitudinal magnetostriction of the specimen, brings the differential-condensor plate (B) to displace and then this displacement unbalances the capacity-bridge. The electric voltage set up by the unbalancing of the capacity-bridge is amplified by the carrier-frequency amplifier and, after detecting, recorded with the magnetizing-current on the same electro-magnetic oscillograph paper, as shown in Fig. 3.

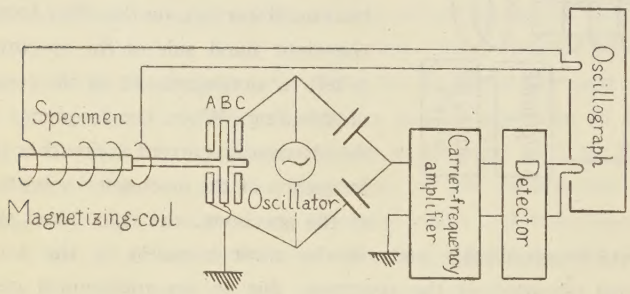

Fig. 2

The intensity of magnetizing field and magnitude of longitudinal magnetostriction are calculated respectively from two curves in Fig. 3.

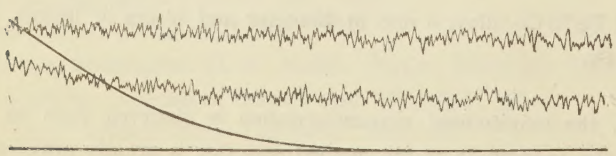

Fig. 3

The intensity of magnetization is also measured ballistically by the method of reversals.

(ii) Specimen.

The specimen is a round bar of Fe-Ni-Co alloy, 6 mm in diameter and 25 cmin length, containing about 45% Ni and 24% Co.

(iii) Result.

At first, this specimen is annealed at 1000°C for 2 hours, baked at 470°C for 20 hours and then cooled comparatively rapidly with the furnace.

As seen from Fig. 3, the longitudinal magnetostriction of the specimen in the initial permeability range is almost in proportion to the intensity of the magnetic field, as in the case of nickel. The I-H curve of the specimen is shown by the curve denoted by (I) in the upper part of the Fig. 4 and the relation between longitudinal magnetostriction and intensity of magnetization is shown in the lower part of the Fig. 4. As mentioned in the previous paper, the magnitude of the longitudinal magnetostricition is not accurat eand contains a considerable quantitative error.

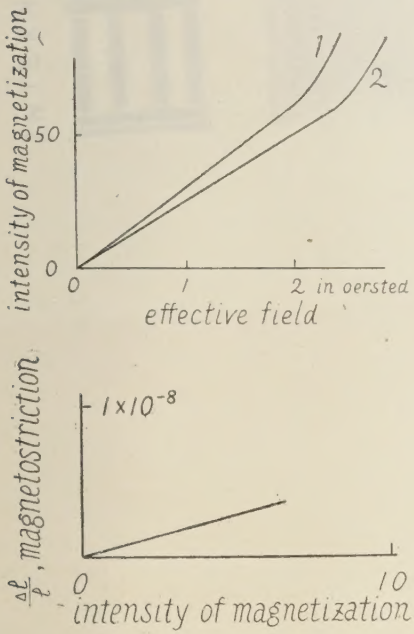

Fig. 4

When this specimen is annealed again at 1000°C for 2 hours, baked at 470°C for 60 hours and then cooled slowly, the longitudinal magnetostriction decreases to be hardly found. The I-H curve in this case is not the same as in the former case and is shown by the curve denoted by (2) in the upper part of the Fig. (4).

The results of the measurement in the static magnetic field show the same tendency as in the alternate magnetic field.

4. Summary.

We take measurement of the longitudinal magnetostriction in the perminvar-alloy. According to the results of the measurement, the longitudinal magnetostriction is observed even in the initial permeability range and is greatly affected by heat-treatment and method of demagnetization, as in the case of nickel.

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